Proof Search and Counter-Model Construction for Bi-intuitionistic Propositional Logic with Labelled Sequents

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Abstract. Bi-intuitionistic logic is a conservative extension of intuitionistic logic with a connective dual to implication, called exclusion. We present a sound and complete cut-free labelled sequent calculus for bi-intuitionistic propositional logic, **BiInt**, following S. Negri's general method for devising sequent calculi for normal modal logics. Although it arises as a natural formalization of the Kripke semantics, it is does not directly support proof search. To describe a proof search procedure, we develop a more algorithmic version that also allows for counter-model extraction from a failed proof attempt.

1 Introduction

Bi-intuitionistic logic (also known as Heyting-Brouwer logic, subtractive logic) is an extension of intuitionistic logic with a connective dual to implication, called exclusion (coimplication, subtraction), a symmetrization of intuitionistic logic. It first got the attention of C. Rauszer [14,15,16], who studied its algebraic and Kripke semantics, alongside adequate Hilbert-style systems and sequent calculi. More recently, it has been of interest to Łukowski [9], Restall [17], Crolard [2] and Goré with colleagues [6,1,7,8]. Part of the motivation is the expected computational significance of the logic: one would expect proof systems working as languages for programming with values and continuations in a symmetric way.

A particularity of bi-intuitionistic logic is that it admits simple sequent calculi obtained from the standard ones for intuitionistic logic essentially by dualizing the rule for implication. Although several authors have stated or "proved" that these calculi enjoy cut elimination (most notably Rauszer [15] for her sequent calculus), they are in fact incomplete without cut and thus not directly suitable for backward (i.e., root-first) proof search. The reasons of the failure are similar to those for the modal logic **S5** (**S4** + symmetry) and the future-past tense logic **KtT4** (**S4** + modalities for the converse of the accessibility relation). A closer analysis suggests that finding remedies that are satisfactory, both from

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the structural proof theory and automated theorem proving points of view, is challenging and provides insights into the subtleties of the logic.

In this paper we propose one solution to the problem. We describe a cut-free labelled sequent calculus for bi-intuitionistic propositional logic, **BiInt**, where the labels are interpreted as worlds in Kripke structures. Exploiting the fact that **BiInt** admits a translation to the future-past tense logic **KtT4**, we obtain it by the general method of S. Negri [12] for devising sequent calculi for normal modal logics. Then, to formulate a search procedure and obtain a termination argument we fine-tune it for the constructive logic situation with monotonicity of truth. This approach is in line with S. Negri's method where frame conditions are uniformly transformed into inference rules, but termination of proof search of the resulting sequent calculus must be obtained on a case-by-case basis. Interestingly, bi-intuitionistic logic turns out to be a rather delicate case.

Cut-free sequent calculi for **BiInt** have also been proposed by Goré and colleagues. Goré's first formulation [6] was in the display logic format, inspired by a general method for devising display systems for normal modal logics. The next formulation by Postniece and Goré [1,7] achieves cut-freedom by combining refutation with proof (passing failure information from premise to premise) to be able to glue counter-models together without the risk of violating the monotonicity condition of interpretations. The new nested sequent calculus by Goré, Postniece and Tiu [8] is a refinement of the display logic version and basically allows reasoning in a local world of a Kripke structure with references to facts about its neighbouring worlds captured in the nested structure.

The paper is organized as follows. In Sect. 2, we introduce **BiInt** with its Kripke semantics and the translation to **KtT4**. We also show its Dragalin-style sequent calculus and why cut elimination fails. In Sect. 3, we introduce a labelled sequent calculus for **BiInt** designed according to S. Negri's recipe. In Sect. 4, we refine this declarative system into a more algorithmic version, show that it is sound and its rules also preserve falsifiability. In the next section (Sect. 5) we define a proof search procedure for the calculus and show that it terminates. In Sect. 6 we put the pieces together to conclude completeness. In the final section we sum up and outline some directions for further enquiry.

2 Bi-intuitionistic Propositional Logic, Dragalin-Style Sequent Calculus and Failure of Cut Elimination

We start by defining the logic **BiInt**. The language extends that of intuitionistic propositional logic, **Int**, by one connective, exclusion, thus the *formulae* are given by the grammar:

$$A, B := p \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid A \prec B$$

where p ranges over a denumerable set of *propositional variables* which give us atoms; the formula $A \prec B$ is the *exclusion* of B from A. We do not take negations as primitive, but in addition to the intuitionistic (or strong) negation, we have dual-intuitionistic (or weak) negation, definable by $\neg A := A \supset \bot$ and $\backsim A := \top \prec A$.

The Kripke semantics defines truth relative to worlds in Kripke structures that are the same as for **Int**. A *Kripke structure* is a triple $K = (W, \leq, I)$ where W is a non-empty set whose elements we think of as *worlds*, $\leq$ is a preorder (reflexive-transitive binary relation) on W (the *accessibility relation*) and I—the *interpretation*—is an assignment of sets of propositional variables to the worlds, which is monotone w.r.t. $\leq$, i.e., whenever $w \leq w'$, we have $I(w) \subseteq I(w')$.

Truth in Kripke structures is defined as for **Int**, but covers also exclusion, interpreted dually to implication as possibility in the past:

- $w \models p$ iff $p \in I(w)$;
- $w \models \top$ always; $w \models \bot$ never;
- $w \models A \wedge B$ iff $w \models A$ and $w \models B$; $w \models A \vee B$ iff $w \models A$ or $w \models B$;
- $w \models B \supset A$ iff, for any $w' \geq w$, $w' \not\models B$ or $w' \models A$;
- $w \models A \prec B$ iff, for some $w' \leq w$, $w' \models A$ and $w' \not\models B$.

A formula is called *valid* if it is true in all worlds of all structures. It is easy to see that monotonicity extends from atoms to all formulae thanks to the universal and existential semantics of implication and exclusion.

It is also a basic observation that the Gödel translation of **Int** into the modal logic **S4** extends to a translation into the future-past tense logic **KtT4** (cf. [9]). As the semantics of **KtT4** does not enforce monotonicity of interpretations, atoms must be translated as future necessities or past possibilities (these are always monotone): $p^{\#} = \Box p$ (or $\blacklozenge p$); $\top^{\#} = \top$; $\bot^{\#} = \bot$; $(A \wedge B)^{\#} = A^{\#} \wedge B^{\#}$; $(A \vee B)^{\#} = A^{\#} \vee B^{\#}$; $(B \supset A)^{\#} = \Box(B^{\#} \supset A^{\#})$; $(A \prec B)^{\#} = \blacklozenge(A^{\#} \prec B^{\#})$.

A sequent calculus for **BiInt** is most easily obtained from Dragalin's sequent calculus for **Int** (as has been done by Restall [17] and Crolard [2]; Rauszer's [15] original sequent calculus was different). In Dragalin's system sequents are multiple-conclusion, but the implication-right rule is constrained. The extension imposes a dual constraint on the exclusion-left rule. The *sequents* are pairs $\Gamma \vdash \Delta$ where Γ, Δ (the *antecedent* and *succedent*) are finite multisets of formulae (we omit braces and denote union by comma as usual). Such a sequent is taken to be *valid* if, for any Kripke structure K and world w, some formula in Γ is false or some formula in Δ is true. The inference rules are displayed in Fig. 1.

Note that the context Δ is missing in the premise of the $\supset R$ rule and dually in the premise of $\prec L$ we do not have the context Γ. The rules $\supset L$ and $\prec R$ involve some contraction. This is necessary because we have chosen not to include a general contraction rule.

This calculus is sound and complete w.r.t. the above-defined notion of validity (completeness can be shown going through the algebraic semantics in terms of Heyting-Brouwer algebras [14]). However it is incomplete without cut, as shown by Pinto and Uustalu in 2003 (private email message from T. Uustalu to R. Goré, 13 Sept. 2004, quoted in [1]). It suffices to consider the obviously valid sequent

initial rule and cut:

$$\frac{}{\Gamma, A \vdash A, \Delta}\ hyp \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ cut$$

logical rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta}\ \top L \qquad \frac{}{\Gamma \vdash \top, \Delta}\ \top R \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge L \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge R$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\ \bot L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ \bot R \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee R$$

$$\frac{\Gamma, B \supset A \vdash B, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma, B \supset A \vdash \Delta}\ \supset L \qquad \frac{\Gamma, B \vdash A}{\Gamma \vdash B \supset A, \Delta}\ \supset R$$

$$\frac{A \vdash B, \Delta}{\Gamma, A \prec B \vdash \Delta}\ \prec L \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash A \prec B, \Delta}{\Gamma \vdash A \prec B, \Delta}\ \prec R$$

Fig. 1. Dragalin-style sequent calculus for **BiInt**

$p \vdash q, r \supset ((p \prec q) \wedge r)$. The only possible last inference in a proof could be

$$\frac{\overset{?}{p, r \vdash (p \prec q) \wedge r}}{p \vdash q, r \supset ((p \prec q) \wedge r)}\ \supset R$$

but the premise is invalid as the succedent formula q has been lost. With cut, the sequent is proved as follows:

$$\frac{\dfrac{\dfrac{}{p \vdash q, p, \ldots}\ hyp \quad \dfrac{}{p, q \vdash q, p \prec q, \ldots}\ hyp}{p \vdash q, p \prec q, \ldots}\ \prec R \qquad \dfrac{\dfrac{\dfrac{}{p, p \prec q, r \vdash p \prec q}\ hyp \quad \dfrac{}{p, p \prec q, r \vdash r}\ hyp}{p, p \prec q, r \vdash (p \prec q) \wedge r}\ \wedge R}{p, p \prec q \vdash q, r \supset ((p \prec q) \wedge r)}\ \supset R}{p \vdash q, r \supset ((p \prec q) \wedge r)}\ cut$$

Cut elimination fails as we cannot permute the cut on the exclusion $p \prec q$ up past the $\supset R$ inference for which the cut formula is a side formula. This is one type of cuts that cannot be eliminated, there are altogether 3 such types [11]. This situation is similar to the naive sequent calculus for **S5** where the sequent $p \vdash \Box\Diamond p$ cannot be proved without cut, but can be proved by applying cut to the sequents $p \vdash \Diamond p$ and $\Diamond p \vdash \Box\Diamond p$ that are provable without cut.

3 L: A Labelled Sequent Calculus

We now proceed to a labelled sequent calculus for bi-intuitionistic logic that we call **L**. This calculus turns out to be complete without a cut rule. Essentially it is a formalization of the first-order theory of the Kripke semantics in such a fashion that the extralogical axioms corresponding to the reflexivity-transitivity condition on frames and monotonicity condition on interpretations do not necessitate cut. Our design follows the method of S. Negri [12].

We proceed from a denumerable set of *labels*. A *labelled formula* is a pair $x : A$ where x is a label and A a formula. The intended meaning is truth of the formula at a particular world.

preorder rules:

$$\frac{\Gamma \vdash_{G\cup\{(x,x)\}} \Delta}{\Gamma \vdash_G \Delta}\ \mathit{refl} \qquad \frac{xGy \quad yGz \quad \Gamma \vdash_{G\cup\{(x,z)\}} \Delta}{\Gamma \vdash_G \Delta}\ \mathit{trans}$$

initial rule and monotonicity rules:

$$\frac{}{\Gamma, x : A \vdash_G x : A, \Delta}\ \mathit{hyp} \qquad \frac{xGy \quad \Gamma, x : A, y : A \vdash_G \Delta}{\Gamma, x : A \vdash_G \Delta}\ \mathit{monL} \qquad \frac{yGx \quad \Gamma \vdash_G y : A, x : A, \Delta}{\Gamma \vdash_G x : A, \Delta}\ \mathit{monR}$$

logical rules:

$$\frac{\Gamma \vdash_G \Delta}{\Gamma, x : \top \vdash_G \Delta}\ \top L \qquad \frac{}{\Gamma \vdash_G x : \top, \Delta}\ \top R \qquad \frac{\Gamma, x : A, x : B \vdash_G \Delta}{\Gamma, x : A \wedge B \vdash_G \Delta}\ \wedge L \qquad \frac{\Gamma \vdash_G x : A, \Delta \quad \Gamma \vdash_G x : B, \Delta}{\Gamma \vdash_G x : A \wedge B, \Delta}\ \wedge R$$

$$\frac{}{\Gamma, x : \bot \vdash_G \Delta}\ \bot L \qquad \frac{\Gamma \vdash_G \Delta}{\Gamma \vdash_G x : \bot, \Delta}\ \bot R \qquad \frac{\Gamma, x : A \vdash_G \Delta \quad \Gamma, x : B \vdash_G \Delta}{\Gamma, x : A \vee B \vdash_G \Delta}\ \vee L \qquad \frac{\Gamma \vdash_G x : A, x : B, \Delta}{\Gamma \vdash_G x : A \vee B, \Delta}\ \vee R$$

$$\frac{xGy \quad \Gamma \vdash_G y : B, \Delta \quad \Gamma, y : A \vdash_G \Delta}{\Gamma, x : B \supset A \vdash_G \Delta}\ \supset L \qquad \frac{y \notin G, \Gamma, \Delta \quad \Gamma, y : B \vdash_{G\cup\{(x,y)\}} y : A, \Delta}{\Gamma \vdash_G x : B \supset A, \Delta}\ \supset R$$

$$\frac{y \notin G, \Gamma, \Delta \quad \Gamma, y : A \vdash_{G\cup\{(y,x)\}} y : B, \Delta}{\Gamma, x : A \prec B \vdash_G \Delta}\ \prec L \qquad \frac{yGx \quad \Gamma \vdash_G y : A, \Delta \quad \Gamma, y : B \vdash_G \Delta}{\Gamma \vdash_G x : A \prec B, \Delta}\ \prec R$$

Fig. 2. Labelled sequent calculus **L**

Sequents are triples $\Gamma \vdash_G \Delta$ where Γ and Δ are finite multisets of labelled formulae, and G is a finite binary relation on labels called the *graph*. Graphs are a means to keep track of label dependencies and thus induce an accessibility relation on worlds.

The inference rules are presented in Fig. 2. Some of them have provisos, that we also write as rule premises. We let xGy abbreviate $(x, y) \in G$. Following usual sequent calculus terminology, at a given rule, we call the explicit labelled formula in the conclusion the *labelled formula introduced* by the rule or the *main labelled formula* of the rule and the explicit labelled formulae in the premises the *side labelled formulae.*

The interesting logical rules are those for implication and exclusion which are dual. Notice the freshness condition on the label y in the rules $\supset R$ and $\prec L$, guaranteeing their soundness. We call label y the *eigenlabel* of the rule and x the *parent* of y. Note also the presence of the monotonicity rules accounting for propagation of truth (resp. falsity) to future (resp. past) worlds and preorder rules which account for reflexivity and transitivity of accessibility.

The counter-example to cut elimination for the Dragalin-style sequent calculus is proved in **L** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : p, y : r \vdash_{(x,y)} x : q, x : p}\ \mathit{hyp} \qquad \dfrac{}{x : p, y : r, x : q \vdash_{(x,y)} x : q}\ \mathit{hyp}}{x : p, y : r \vdash_{(x,y)} x : q, y : p \prec q}\ \prec R \qquad \dfrac{}{x : p, y : r \vdash_{(x,y)} x : q, y : r}\ \mathit{hyp}}{x : p, y : r \vdash_{(x,y)} x : q, y : (p \prec q) \wedge r}\ \wedge R}{x : p \vdash_{\emptyset} x : q, x : r \supset ((p \prec q) \wedge r)}\ \supset R$$

Notice the downward information propagation in the $\prec R$ inference to an already existing label.

In a **L**-derivation the names of the eigenlabels can be changed (to new names not occurring in the derivation) without changing the end sequent. This property allows us to show by usual methods that **L** enjoys admissibility of the weakening rules. A simple combination of the monotonicity, reflexivity and weakening also guarantees admissibility of the contraction rules in **L**. (This is what enables us to avoid explicit contractions at $\supset L$ and $\prec R$ rules.)

The cut rule is also admissible in **L**. This can be proved along the lines of cut elimination results of S. Negri for labelled sequent calculi for modal logics. In this paper, as an immediate consequence of soundness and completeness of system **L** w.r.t. the Kripke semantics (Cor. 1), we get a semantical proof of admissibility of cut.

Given a Kripke structure K, a *K-valuation* is a mapping from the set of labels to the set of worlds of K.

Definition 1. *A Kripke structure $K = (W, \leq, I)$ and a K-valuation v are a* counter-model *(cm) to an* **L***-sequent $\Gamma \vdash_G \Delta$, if: i) for all xGy, $v(x) \leq v(y)$; ii) for all $x : A \in \Gamma$, $v(x) \models A$; and iii) for all $x : A \in \Delta$, $v(x) \not\models A$. The sequent is* valid*, if it has no counter-model.*

Proposition 1 (Soundness of L). *If $\Gamma \vdash_G \Delta$ is derivable, $\Gamma \vdash_G \Delta$ is valid.*

Completeness holds as well (Cor. 1) and is proved with the help of the algorithmic version of **L** introduced in the next section. In fact our completeness argument allows for construction of counter-models of non-derivable sequents.

4 L*: An Algorithmic Version of L

Although **L** constitutes a good basis for backward proof search for bi-intuitionistic propositional logic, it still faces the problem that the preorder and monotonicity rules can be applied at any point in backward proof search. To deal with this problem, we introduce now an algorithmic version of **L** called **L***. System **L*** does not have explicit preorder or monotonicity rules. It uses a marking mechanism on certain kinds of labelled formulae. Such mechanism allows for the recovering of labelled formulae, so that monotonicity requirements are guaranteed. The marking mechanism is also designed in a way that it can be used in loop-detection, to avoid infinite search along paths corresponding to non-derivable sequents.

Sequents in **L*** are triples $\Gamma \vdash_G \Delta$ as in **L**, with the difference that, in the contexts Γ and Δ, labelled formulae can now be marked either with $*$ (written as $x : A^*$), $\circ$ (written as $x : A^\circ$) or with $\bullet$ (written as $x : A^\bullet$). The rules of **L*** are in Fig. 3.

Let us briefly explain the role of $^+$ and $-$ and of marks $*$, $\circ$ and $\bullet$ in backward proof search. The $^+$ (resp. $-$) is used to propagate a formula to future (resp. past) labels (as determined by the transitive closure of the graph). The marking

initial rule:

$$\frac{}{\Gamma, x : p^\circ \vdash_G x : p^\circ, \Delta}\ hyp$$

atom rules:

$$\frac{\Gamma, p^+, x : p^*, x : p^\circ \vdash_G \Delta}{\Gamma, x : p \vdash_G \Delta}\ atomL \qquad \frac{\Gamma \vdash_G x : p^\circ, x : p^*, p^-, \Delta}{\Gamma \vdash_G x : p, \Delta}\ atomR$$

where $p^+ = \{y : p \mid xGy\}$ where $p^- = \{y : p \mid yGx\}$

logical rules:

$$\frac{\Gamma \vdash_G \Delta}{\Gamma, x : \top \vdash_G \Delta}\ \top L \quad \frac{}{\Gamma \vdash_G x : \top, \Delta}\ \top R \quad \frac{\Gamma, x : A, x : B \vdash_G \Delta}{\Gamma, x : A \wedge B \vdash_G \Delta}\ \wedge L \quad \frac{\Gamma \vdash_G x : A, \Delta \quad \Gamma \vdash_G x : B, \Delta}{\Gamma \vdash_G x : A \wedge B, \Delta}\ \wedge R$$

$$\frac{}{\Gamma, x : \bot \vdash_G \Delta}\ \bot L \quad \frac{\Gamma \vdash_G \Delta}{\Gamma \vdash_G x : \bot, \Delta}\ \bot R \quad \frac{\Gamma, x : A \vdash_G \Delta \quad \Gamma, x : B \vdash_G \Delta}{\Gamma, x : A \vee B \vdash_G \Delta}\ \vee L \quad \frac{\Gamma \vdash_G x : A, x : B, \Delta}{\Gamma \vdash_G x : A \vee B, \Delta}\ \vee R$$

$$\frac{\Gamma, (B \supset A)^+, x : (B \supset A)^* \vdash_G x : B, \Delta \quad \Gamma, x : A \vdash_G \Delta}{\Gamma, x : B \supset A \vdash_G \Delta}\ \supset L$$

where $(B \supset A)^+ = \{y : B \supset A \mid xGy\}$

$$\frac{x : (B \supset A)^\bullet \notin \Delta \quad y \notin G, \Gamma, \Delta, \quad \Gamma, \Gamma^{y/x}, y : B \vdash_{G \cup \{(x,y)\}} y : A, x : (B \supset A)^\bullet, \Delta}{\Gamma \vdash_G x : B \supset A, \Delta}\ \supset R$$

where $\Gamma^{y/x} = \{y : C \mid x : C^* \in \Gamma\} \cup \{y : p^\circ \mid x : p^\circ \in \Gamma\}$
$\cup \{y : (C \prec D)^\bullet \mid x : C \prec D \in \Gamma \text{ or } x : (C \prec D)^\bullet \in \Gamma\}$

$$\frac{x : (A \prec B)^\bullet \notin \Gamma \quad y \notin G, \Gamma, \Delta \quad \Gamma, x : (A \prec B)^\bullet, y : A \vdash_{G \cup \{(y,x)\}} y : B, \Delta^{y/x}, \Delta}{\Gamma, x : A \prec B \vdash_G \Delta}\ \prec L$$

where $\Delta^{y/x} = \{y : C \mid x : C^* \in \Delta\} \cup \{y : p^\circ \mid x : p^\circ \in \Delta\}$
$\cup \{y : (D \supset C)^\bullet \mid x : D \supset C \in \Delta \text{ or } x : (D \supset C)^\bullet \in \Delta\}$

$$\frac{\Gamma \vdash_G x : A, \Delta \quad \Gamma, x : B \vdash_G x : (A \prec B)^*, (A \prec B)^-, \Delta}{\Gamma \vdash_G x : A \prec B, \Delta}\ \prec R$$

where $(A \prec B)^- = \{y : A \prec B \mid yGx\}$

Fig. 3. Algorithmic version $\mathbf{L}^*$

$x : A^*$ is done at the atom rules, $\supset L$ and $\prec R$ (where $x : A$ is the main formula) in order to be able to recover A at eventual labels still unknown when $x : A$ is analysed, but later created with a graph connection to x. The marking of a labelled formula with a $\circ$ (used only with atoms) or $\bullet$ (used only with implications and exclusions) means essentially that the formula was already analysed (the case with the explicit circles and bullets in the rule premises of the atom rules, $\supset R$, $\prec L$) or has no further useful information and so need not be analysed (the case with circles and bullets implicit in $\Gamma^{y/x}$ and $\Delta^{y/x}$ in the premises of $\supset R$ and $\prec L$ respectively) and prevents a new analysis of the formula at the given world (notice that no rule introduces a labelled formula with a circle or a bullet).

Notice that an **L**-sequent is also an $\mathbf{L}^*$-sequent and that if we take an $\mathbf{L}^*$-sequent and erase all $*$, $\circ$ and $\bullet$ marks we obtain an **L**-sequent. Given an $\mathbf{L}^*$

context Γ, we write Γ^- for the **L**-context resulting from it by replacing all labelled formulae $x : A^*$, $x : A^\circ$ with unmarked labelled formulae $x : A$ and removing all labelled formulae $x : A^\bullet$. Given an $\mathbf{L}^*$-sequent $\Gamma \vdash_G \Delta$ its *erasure* is the **L**-sequent $\Gamma^- \vdash_G \Delta^-$. We say that an $\mathbf{L}^*$-rule is derivable in **L** if the rule obtained by replacing its premises and conclusion by their erasures is derivable in **L**. The next proposition shows that all $\mathbf{L}^*$-rules are derivable in **L** and thus $\mathbf{L}^*$ is sound w.r.t. **L**.

Proposition 2 (Soundness of $\mathbf{L}^*$ w.r.t. L). *1. All rules of* $\mathbf{L}^*$ *are derivable in* **L**. *2. If* $\Gamma \vdash_G \Delta$ *is derivable in* $\mathbf{L}^*$ *then* $\Gamma^- \vdash_G \Delta^-$ *is derivable in* **L**.

Because the rules of $\mathbf{L}^*$ do not throw away any relevant information (read backward, i.e., from the conclusion to the premises), they have the strong property that a counter-model of a premise is also a counter-model of the conclusion. This is used in Sec. 6 for extracting counter-models out of failed proof attempts.

Given a Kripke structure $K = (W, \leq, I)$, a K-valuation v and a graph G, $\leq_G^-$ denotes the relation $\leq \setminus v(G)^*$, i.e., $\leq_G^-$ is the relation obtained from $\leq$ by eliminating all pairs in the reflexive-transitive closure of $\{(v(x), v(y)) \mid xGy\}$.

Definition 2. *A Kripke structure* $K = (W, \leq, I)$ *and a* K*-valuation* v *are a* counter-model *of an* $\mathbf{L}^*$*-sequent* $\Gamma \vdash_G \Delta$ *when:*

1. *for all* xGy, $v(x) \leq v(y)$;
2. *for all* $x : A, x : A^\circ \in \Gamma$, $v(x) \models A$;
3. *for all* $x : A^* \in \Gamma$ *and for all* $w \in W$ *such that* $v(x) \leq_G^- w$, $w \models A$;
4. *for all* $x : A, x : A^\circ \in \Delta$, $v(x) \not\models A$;
5. *for all* $x : A^* \in \Delta$ *and for all* $w \in W$ *such that* $w \leq_G^- v(x)$, $w \not\models A$.

Notice that for **L**-sequents this notion of counter-model coincides with the notion introduced in the previous section. As usual *valid sequents* are those for which there are no counter-models.

Proposition 3 (Preservation of counter-models). *For each* $\mathbf{L}^*$*-rule, a counter-model of a premise is also a counter-model of the conclusion.*

5 A Search Procedure and Its Termination

We now describe a backward search procedure for $\mathbf{L}^*$, which incorporates a loop-checking mechanism, and prove it sound and terminating. As a by-product of the explicit presence in sequents of labels/worlds and the graph/accessibility relation, when the search procedure terminates with failure, we will be left with a Kripke counter-model of the given sequent. This fact is proved in the next section and accounts for the completeness of the search procedure. In order to describe the search procedure, we introduce first some terminology, notation and also the loop-rules.

The rules $\supset R$ and $\prec L$ are the only rules of $\mathbf{L}^*$ where the graph relation varies in a backward reading. We call these rules *world creating rules.* A sequent

$$\frac{y \notin G \quad \Gamma \setminus \Gamma(y) \vdash_G \Delta[x/y]}{\Gamma \vdash_{G \cup \{(x,y)\}} \Delta}\ loopUp \qquad \frac{y \notin G \quad \Gamma[x/y] \vdash_G \Delta \setminus \Delta(y)}{\Gamma \vdash_{G \cup \{(y,x)\}} \Delta}\ loopDn$$

provided $\Gamma[y] \subseteq \Gamma[x] \cup \Gamma^\bullet[x], \Gamma^*[y] \subseteq \Gamma^*[x]$, and $\Gamma^\circ[y] \subseteq \Gamma^\circ[x]$ provided $\Delta[y] \subseteq \Delta[x] \cup \Delta^\bullet[x], \Delta^*[y] \subseteq \Delta^*[x]$, and $\Delta^\circ[y] \subseteq \Delta^\circ[x]$

Fig. 4. Loop rules

$\Gamma \vdash_G \Delta$ is called *saturated* if it is irreducible w.r.t. the non-world creating rules. A sequent $\Gamma \vdash_G \Delta$ is called *stuck* when it is irreducible w.r.t. any rule and moreover it is not an *axiom* (hyp, $\bot L$, $\top R$).

Given an $\mathbf{L}^*$ context Γ, we use the notations $\Gamma[x]$, $\Gamma^*[x]$, $\Gamma^\circ[x]$, $\Gamma^\bullet[x]$ and $\Gamma(x)$ to mean $\{A \mid x{:}A \in \Gamma\}$, $\{A \mid x{:}A^* \in \Gamma\}$, $\{A \mid x{:}A^\circ \in \Gamma\}$, $\{A \mid x{:}A^\bullet \in \Gamma\}$ and $\Gamma[x] \cup \Gamma^*[x] \cup \Gamma^\circ[x] \cup \Gamma^\bullet[x]$ respectively.

The *loop rules* are presented in Fig. 4. (For a context Γ and labels x and y, the notation $\Gamma[x/y]$ stands for the context obtained by replacing y with x in Γ.) Their backward reading corresponds to the action taken when a loop is detected and the detection of a loop corresponds to satisfaction of their side-conditions. The formulation of the loop rules corresponds to the situation where x is the parent and thus y is a descendant of x, labeling necessarily subformulae of x-labelled formulae.

We are now in conditions of presenting the *search procedure*. It goes as follows:

1. Given an $\mathbf{L}^*$-sequent $\Gamma \vdash_G \Delta$, we reduce it w.r.t. the non-world creating rules (i.e., we apply as long as possible these rules). We call a *saturation* both this process and the partial proof of $\Gamma \vdash_G \Delta$ so constructed. The top sequent of each branch of a saturation is a *saturated sequent*. Notice also that the order in which rules are applied in saturation is unimportant since they are inter-permutable.
2. Then, for every branch in the saturation of $\Gamma \vdash_G \Delta$, we do the following:
   (a) we check if the top sequent is an axiom and if so search along the branch is stopped with *success*;
   (b) we check if there is a loop, i.e., we test if the side condition of any of the loop rules is met, and if so proceed according to the corresponding loop rule.
3. If neither (a) nor (b) is the case, the development of the branch carries on, by applying one of the world creating rules, and we go back to 1.We stop with *failure* if no world creating rule can be applied.

We call *proof attempt* both the run of the search procedure with a given sequent and the corresponding partial proof (in $\mathbf{L}^*$ augmented with the loop rules). Throughout we assume that proof attempts always start with $\mathbf{L}$-sequents whose graphs are trees (i.e., the graph, seen as an undirected graph by forgetting the directions of the arcs, is connected and acyclic). Then the graphs of all sequent in the proof attempt are trees.

Proposition 4 (Soundness of the search procedure). *If the proof attempt for an* $\mathbf{L}$*-sequent terminates with success, then the sequent is* $\mathbf{L}$*-derivable.*

Proof: By induction on the height of the proof attempt, we prove that, for any $\mathbf{L}^*$-sequent $\Gamma \vdash_G \Delta$ in it, $\Gamma^- \vdash_G \Delta^-$ is derivable in $\mathbf{L}$. The cases corresponding to $\mathbf{L}^*$-inferences follow by part 1. of Prop. 2. Consider the case corresponding to the *loopUp* rule of Fig. 4. (The case of *loopDn* is similar.) By IH we have that $(\Gamma \setminus \Gamma(y))^- \vdash_G \Delta[x/y]^-$ is $\mathbf{L}$-derivable. From this, by weakening , $\Gamma^- \vdash_{G\cup\{(x,y)\}} (\Delta \setminus \Delta(y))^-, \Delta(y)[x/y]^-, \Delta(y)^-$is also derivable in $\mathbf{L}$. Since xGy, by repeated use of *monR*, we can derive $\Gamma^- \vdash_{G\cup\{(x,y)\}} (\Delta \setminus \Delta(y))^-, \Delta(y)^-$which is $\Gamma^- \vdash_{G\cup\{(x,y)\}} \Delta^-$. □

Now we consider terminology, notation and lemmata used in particular for proving termination of the search procedure. Given a label x, the *world creation tree* of x induced by a branch of a proof attempt has x as root and has as subtrees (if any) the world creation trees of each eigenlabel in the branch whose parent is x. Given a set of formulas S, $\mathsf{mhf}(S)$ stands for the maximal height of the formulae in S.

Lemma 1. *Any saturation in a proof attempt is finite.*

Proof: Observe that: (i) $\wedge$ and $\vee$ inferences replace the main formula by strict subformulae; and (ii) even if the main formula of an *atomL*, *atomR*, $\supset L$ or $\prec R$ inference may reappear in the premises or upper sequents, it does so with a distinct label (as the graph is a tree) and thus can only reappear finitely many times (recall $\mathbf{L}^*$-graphs are finite). □

Lemma 2. *Given a label x and a branch $\mathcal{B}$ of a proof attempt, x has finitely many children in $\mathcal{B}$.*

Proof: Notice that all formulae in a sequent of $\mathcal{B}$ are subformulae of a formula in the end sequent of $\mathcal{B}$ (which is finite) and that, once $x : A \supset B$ (resp. $x : A \prec B$) is analysed as the main formula of a $\supset R$ (resp. $\prec L$) inference, $x : (A \supset B)^\bullet$ (resp. $x : (A \prec B)^\bullet$) is added to the succedent (resp. antecedent) of the inference's premise, preventing that $x : A \supset B$ (resp. $x : A \prec B$) becomes analysed again. □

Lemma 3. *For $\diamond \in \{*, \circ\}$ and any saturated sequent $\Gamma \vdash_{G\cup\{(x,y)\}} \Delta$ in a proof attempt: i) if $x : A^\diamond \in \Gamma$, $y : A^\diamond \in \Gamma$; and ii) if $y : A^\diamond \in \Delta$, $x : A^\diamond \in \Delta$.*

Proof: Firstly notice that, for any $\mathbf{L}^*$-rule, if $z : B^\diamond$ is in the antecedent (resp. succedent) of the conclusion, $z : B^\diamond$ is in the antecedent (resp. succedent) of any premise. Consider the case $x : p^\circ \in \Gamma$ (the other cases being similar or simpler). Then $x : p^* \in \Gamma$ and thus $x : p$ must have been the main formula in an *atomL* inference. Let $\Gamma_0 \vdash_{G_0} \Delta_0$ be the premise of that inference. If $(x, y) \in G_0$, $y : p \in \Gamma_0$ and any top sequent in the saturation of $\Gamma_0 \vdash_{G_0} \Delta_0$ has both $y : p^*$ and $y : p^\circ$ in the antecedent. If not, above the referred inference, there must be an $\supset R$ inference with eigenlabel y and parent x and the top sequents of the saturation of its premise have $y : p^*$ and $y : p^\circ$ in their antecedents. □

Lemma 4. *In a proof attempt, if $\Gamma_0 \vdash_G \Delta_0$ is the conclusion of an $\supset R$ inference with eigenlabel x_1 and parent x_0 and $\Gamma_1 \vdash_{G\cup\{(x_0,x_1)\}} \Delta_1$ is a top sequent in the saturation of the inference's premise, then $\Gamma_0(x_0) \subset \Gamma_1(x_1)$.*

Proof: By the following three facts: i) $\Gamma_0(x_0) \subseteq \Gamma_1(x_0)$, because no $\mathbf{L}^*$ rule removes starred, circled or bulleted formulae (when read backwards); ii) $\Gamma_1(x_0) \subseteq \Gamma_1(x_1)$, because $\Gamma_1[x_0] \cup \Gamma_1^\bullet[x_0] \subseteq \Gamma_1^\bullet[x_1]$, $\Gamma_1^*[x_0] \subseteq \Gamma_1^*[x_1]$, $\Gamma_1^\circ[x_0] \subseteq \Gamma_1^\circ[x_1]$ (the last two containments proved with the help of Lemma 3); iii) $\Gamma_1(x_1) \not\subseteq \Gamma_1(x_0)$, because of the loop checking mechanism. □

Lemma 5. *For any sub-branch of a proof attempt of the form*

$$\dfrac{\dfrac{\dfrac{\Gamma_2 \vdash_{G_1 \cup \{(x_2,x_1)\}} \Delta_2}{\vdots}}{\dfrac{\Gamma_1, x_1 : (A_1 \prec B_1)^\bullet, x_2 : A_1 \vdash_{G_1 \cup \{(x_2,x_1)\}} x_2 : B_1, \Delta_1, \Delta_1^{x_2/x_1}}{\Gamma_1, x_1 : A_1 \prec B_1 \vdash_{G_1} \Delta_1} \prec L}}{\vdots}$$

$$\dfrac{\Gamma_0, \Gamma_0^{x_1/x_0}, x_1 : A_0 \vdash_{G_0 \cup \{(x_0,x_1)\}} x_1 : B_0, x_0 : (A_0 \supset B_0)^\bullet, \Delta_0}{\Gamma_0 \vdash_{G_0} x_0 : A_0 \supset B_0, \Delta_0} \supset R$$

where the conclusion of $\prec L$ *is a top sequent in the saturation of the premise of* $\supset R$ *and* $\Gamma_2 \vdash_{G_1 \cup \{(x_2,x_1)\}} \Delta_2$ *is a top sequent in the saturation of the premise of* $\prec L$ *we have* $\mathsf{mhf}(\Gamma_0(x_0) \cup \{A_0 \supset B_0\}) > \mathsf{mhf}(\Gamma_2(x_2) \cup \Delta_2(x_2))$.

Proof: By the following two facts: i) each formula of $\Gamma_2(x_2) \cup \Delta_2(x_2)$ is a subformula of $\Delta_1(x_1)$ or a strict subformula of $A_1 \prec B_1$; and ii) $\Delta_1(x_1)$ has only strict subformulae of $\Gamma_0(x_0) \cup \{A_0 \supset B_0\}$ and $A_1 \prec B_1$ is a subformula of $\Gamma_0(x_0) \cup \{A_0 \supset B_0\}$[1]. □

Theorem 1 (Termination of the search procedure). *A proof attempt always terminates.*

Proof: If it did not, by König's Lemma there would be an infinite branch $\mathcal{B}$ in the proof attempt. Since each saturation is finite (Lemma 1), there must be infinitely many saturations and at least one of the labels in the end sequent, x_{00} say, has an infinite world creation tree, call it $\mathcal{T}$. By Lemma 2, $\mathcal{T}$ is finitely branching and so König's Lemma forces $\mathcal{T}$ to have an infinite branch, which we will show to be impossible.

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[1] The first fact follows from the following lemma (and the second fact from an analogous lemma): For any sequent $\Gamma \vdash_{G_1 \cup \{(x_2,x_1)\}} \Delta$ in the saturation of the premise of an $\prec L$ inference with eigenlabel x_2 and parent x_1:

1. if $x : A \in \Gamma$ (resp. Δ) and A is not an exclusion (resp. implication), then $x_2(G_1 \cup \{(x_2,x_1)\})^* x$ (resp. $x(G_1 \cup \{(x_2,x_1)\})^* x_2$);
2. if $A \in \Gamma[x_1]$ (resp. $\Delta[x_1]$), then A is an exclusion (resp. implication) or $A \in \Gamma(x_2)$ (resp. $\Delta(x_2)$);
3. if $\Gamma_0 \vdash_{G_1 \cup \{(x_2,x_1)\}} \Delta_0$ is a sequent immediately above $\Gamma \vdash_{G_1 \cup \{(x_2,x_1)\}} \Delta$ and $A \in \Gamma_0(x_2)$ (resp. $\Delta_0(x_2)$), then A is a subformula of $\Gamma(x_2)$ (resp. $\Delta(x_2)$) or a strict subformula of $\Delta(x_2)$ (resp. $\Gamma(x_2)$).

It is impossible that an infinite branch of $\mathcal{T}$ beyond a certain point, z_0 say, goes always upwards, i.e., that the descendants $z_1, z_2, \ldots$ of z_0 in $\mathcal{T}$ all arise with $\supset R$ inferences. Otherwise, using Lemma 4, we could form an infinite sequence

$$\Gamma_0(z_0) \subset \Gamma_1(z_1) \subset \Gamma_2(z_2) \ldots$$

(where Γ_i is the conclusion's antecedent of the inference where z_i creates z_{i+1}), which is is impossible, for all these sets must be included in the finite set of subformulae of the end sequent. Similarly, $\mathcal{T}$ cannot have an infinite branch that beyond a certain point goes always downwards.

Therefore, an infinite branch of $\mathcal{T}$ would have to correspond to an infinite zigzag of the shape shown in Fig. 5 (or of a dual shape), where a dashed arrow up (resp. down) means that zero or more worlds were created by $\supset R$ (resp. $\prec L$) inferences in between x_{i0} and x_{in_i} and a solid arrow up (resp. down) means that $x_{(i+1)0}$ was created from x_{in_i} by an $\supset R$ (resp. $\prec L$) inference. Let Γ_{ij} (resp. Δ_{ij}) stand for the conclusion's antecedent (resp. succedent) of the inference where x_{ij} creates its immediate successor in branch $\mathcal{B}$. By Lemma 5, a property analogous to it (for the case where $\prec L$ is below $\supset R$), and the fact that, given i and $j_1 < j_2 \leq n_i$, $\mathsf{mhf}(\Gamma_{ij_1}(x_{ij_1}) \cup \Delta_{ij_1}(x_{ij_1})) \geq \mathsf{mhf}(\Gamma_{ij_2}(x_{ij_2}) \cup \Delta_{ij_2}(x_{ij_2}))$, it follows that $\mathsf{mhf}(\Gamma_{in_i}(x_{in_i}) \cup \Delta_{in_i}(x_{in_i})) > \mathsf{mhf}(\Gamma_{(i+1)n_{i+1}}(x_{(i+1)n_{i+1}}) \cup \Delta_{(i+1)n_{i+1}}(x_{(i+1)n_{i+1}}))$ and so an infinite descending chain of natural numbers would be produced. □

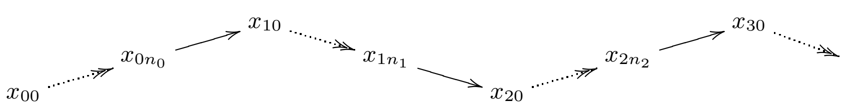

Fig. 5. An infinite zigzag

6 Completeness and Counter-Model Construction

We prove here that when the search procedure introduced in the previous section arrives at a stuck sequent, we can immediately read off from the sequent a Kripke counter-model for it. This result is then instrumental in achieving the equivalence between derivability in **L** and validity.

Theorem 2 (Counter-models at stuck sequents). *Let $\mathcal{B}$ be a failed branch of a proof attempt with top sequent $\Gamma \vdash_G \Delta$.*

1. *The structure $K = (W, \leq, I)$ where W is the set of labels in the sequent, $\leq$ is the reflexive-transitive closure of G and $I(x) = \{p \mid x : p^\circ \in \Gamma\}$, is a Kripke structure.*
2. *Let G_{ext} stand for G extended with all pairs removed in loop steps of $\mathcal{B}$[2]. Let v be the valuation on W such that $v(y) = x$ if (x, y) or (y, x) is in G_{ext} and*

[2] Here is assumed that when an eigenlabel is created at a branch of a proof attempt it is distinct of any other label occurring below in the branch.

$y \notin G$; and $v(y) = y$ otherwise. For any sequent $\Gamma' \vdash_{G'} \Delta'$ in $\mathcal{B}$, (i) for all $xG'y$, $v(x) \leq v(y)$, and moreover (ii) for any formula A,

(a) if $x : A$ or $x : A^\circ$ or $x : A^\bullet$ belongs to Γ', $v(x) \models A$;

(b) if $x : A^* \in \Gamma'$, for all $w \in W$ s.t. $v(x) \leq^-_{G'} w$, $w \models A$;

(c) if $x : A$ or $x : A^\circ$ or $x : A^\bullet$ belongs to Δ', $v(x) \not\models A$;

(d) if $x : A^* \in \Delta'$, for all $w \in W$ s.t. $w \leq^-_{G'} v(x)$, $w \not\models A$;

hence, in particular, (K, v) is a cm of $\Gamma' \vdash_{G'} \Delta'$.

Proof: 1. If $x = x_1Gx_2G...Gx_n = y$ and $x : p^\circ \in \Gamma$, follows by induction on n that $y : p^\circ \in \Gamma$ (using Lemma 3 in the step case).

2. (i) Trivial. (ii) By induction on the formula A and sub-induction on the number of sequents above $\Gamma' \vdash_{G'} \Delta'$ in $\mathcal{B}$. If $\Gamma' \vdash_{G'} \Delta'$ is $\Gamma \vdash_G \Delta$ itself, it can only have circled atoms or else starred or bulleted formulas. The conditions on circled atoms hold by construction of (K, v) and the conditions on starred formulas hold trivially, since $\leq^-_G = \leq \setminus v(G)^* = \emptyset$. If $x : (C \prec D)^\bullet \in \Gamma'$ (for $x : (C \supset D)^\bullet \in \Delta'$ the argument is analogous), it can be proved that there are labels y and z, such that $zG_{\mathsf{ext}}y(G_{\mathsf{ext}})^*x$ and $\mathcal{B}$ includes a step

$$\frac{\Gamma'', y : (C \prec D)^\bullet, z : C \vdash_{G'' \cup \{(z,y)\}} z : D, \Delta''^{z/y}, \Delta''}{\Gamma'', y : C \prec D \vdash_{G''} \Delta''} \prec L$$

By the outer IH, $v(z) \models C$ and $v(z) \not\models D$. Thus, since $zG_{\mathsf{ext}}y(G_{\mathsf{ext}})^*x$ implies $v(z) \leq v(y) \leq v(x)$, $v(x) \models C \prec D$.

If $\Gamma' \vdash_{G'} \Delta'$ is the conclusion of an atom or logical inference, the desired conditions follow by the inner IH applied to the premise in $\mathcal{B}$. Let us consider the case of the *loopUp* rule (*loopDn* is analogous):

$$\frac{y \notin G_0 \quad \Gamma' \setminus \Gamma'(y) \vdash_{G_0} \Delta'[x_0/y]}{\Gamma' \vdash_{G_0 \cup \{(x_0,y)\}} \Delta'} \; loopUp$$
$$\text{provided } \Gamma'[y] \subseteq \Gamma'[x_0] \cup \Gamma'^\bullet[x_0], \Gamma'^*[y] \subseteq \Gamma'^*[x_0], \Gamma'^\circ[y] \subseteq \Gamma'^\circ[x_0]$$

Conditions (a) and (b) restricted to $\Gamma' \setminus \Gamma'(y)$ hold by the inner IH. As to $\Gamma'(y)$: for $y : A^\diamond \in \Gamma'$ ($\diamond \in \{*, \circ\}$), the proviso guarantees $x_0 : A^\diamond \in \Gamma'$ and so, by the inner IH and $v(y) = v(x_0)$, $v(y) \models A$; for $y : A \in \Gamma'$, the proviso guarantees either $x_0 : A \in \Gamma'$ or $x_0 : A^\bullet \in \Gamma'$, but both cases follow also from the inner IH[3]. Conditions (c) and (d) follow from the inner IH and the facts $\Delta'(y) \subseteq (\Delta'[x_0/y])(x)$ and $v(y) = v(x_0)$. □

Corollary 1. *1. Let $\Gamma \vdash_G \Delta$ be an* **L***-sequent whose graph is a tree. The following statements are equivalent: i) $\Gamma \vdash_G \Delta$ is derivable in* **L***; ii) $\Gamma \vdash_G \Delta$ is valid; iii) the attempt to prove $\Gamma \vdash_G \Delta$ terminates with success.*

2. For any **L***-sequent whose graph is a tree, the search procedure yields either a proof or a counter-model.*

[3] The second case illustrates why the inductive argument does not go through, if we simply prove that (K, v) is a cm of $\Gamma' \vdash_{G'} \Delta'$.

Proof: 1. Follows from Thm. 2 with the help of Thm. 1 and Prop. 1.

2. Apply the search procedure to the given sequent. Thm. 1 guarantees that it terminates. If this happens with success, then by Prop. 4 the sequent is provable in **L**. Otherwise, the proof attempt has at least one failed branch and thus Thm. 2 guarantees the wanted cm. □

7 Conclusion

Although bi-intuitionistic logic may seem to be a modest extension of intuitionistic logic, it has proved to be rather intricate from the structural proof theory point of view. While naive sequent calculus formalizations are incomplete without a cut rule, more considerate attempts at the design of sequent calculi for backward proof search seem all to lead to relatively sophisticated designs.

We believe that our labelled sequent calculus represents a meaningful compromise between declarativeness and algorithmicity by encoding a reasonably straightforward Kripke semantics based search strategy very much in the spirit of analytic tableaux. Some novelties include integration of all useful monotonicity consequences into the logical rules, including a specific annotation to deal with consequences that must be delayed (flow of information into worlds not yet created), and a termination argument utilizing the fact that information cannot flow around too many turns. The failure-collecting sequent calculus by Postniece and Goré [1,7] and the new calculus of nested sequents by Goré et al. [8] are systems with the same aim and we find the nested sequent calculus especially neat proof-theoretically, although it may require fine-tuning to be practical in theorem proving/counter-model building.

As future work, we would like to see whether bi-intuitionistic logic admits a loop-free backward-search proof system à la Dyckhoff [4], possibly modifiable into a refutation system [13]. We would like to see if it is possible to devise a system with controlled ("analytic") cuts by a careful analysis of the failure of cut elimination for the Dragalin-style sequent calculus. A yet further line would be to devise a sequent calculus for forward search (a calculus of Mints-style resolution) [10].

On a different note, we would also very much like to come to an understanding of the computational significance of bi-intuitionistic logic, i.e., whether it admits useful a Curry-Howard interpretation justified by a well-motivated, non-degenerate categorical semantics. The first step in this direction was made already by Filinski [5] and further considerations appear in the work of Curien and Herbelin [3]. Crolard's project [2] clearly had the same ultimate aim. We expect that the nested sequences technique of Goré et al. [8] can point to the right structures.

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