Strong Relative Monads (Extended Abstract)

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In a paper presented here in Paphos at FoSSaCS [1], we introduced a generalization of monads, called relative monads, to analyze monad-like structures where the underlying functor is not an endofunctor. This study was motivated by several examples of direct relevance for programming theory, such as finite-dimensional vector spaces, the syntaxes of untyped and simply-typed lambda-calculus, but also the arrow types of Hughes [3], as mathematized by Jacobs et al. [4] (in their weak version without the strength). We showed that the Kleisli and Eilenberg-Moore constructions generalize to relative monads and are related to relative adjunctions (with the Kleisli category of an arrow type as a relative monad turning out to be the same as a Freyd category in the sense of Power and Robinson [6]—much as in the work of Jacobs et al.). We also showed that, under reasonable assumptions, relative monads are monoids in the functor category concerned and extend to ordinary monads, giving rise to a coreflection between relative monads and ordinary monads.

But in programming theory applications, we are often interested in strong monads rather than just monads and the standard concept of an arrow type is also that of a strong arrow type. It is therefore natural to ask whether relativization is possible also for strong monads.

In this paper, we study this question and answer it in the affirmative. We introduce the concept of a strong relative monad on a monoidal functor, show that strong arrow types are the same as strong relative monads on the Yoneda embedding (seen as a monoidal functor) and investigate the properties of strong relative monads.

Given a functor J between two categories $\mathbb{J}$ and $\mathbb{C}$, a relative monad on J is given by: (i) an object function $T \in |\mathbb{J}| \to |\mathbb{C}|$, (ii) for any $X \in |\mathbb{J}|$, a map $\eta_X \in \mathbb{C}(JX, TX)$, and (iii) for any $X, Y \in |\mathbb{J}|$ and $k \in \mathbb{C}(JX, TY)$, a map $k^* \in \mathbb{C}(TX, TY)$, satisfying (i) for any $X, Y \in |\mathbb{J}|$, $k \in \mathbb{C}(JX, TY)$, $k = k^* \circ \eta_X$, (ii) for any $X \in |\mathbb{J}|$, $\eta_X^* = \mathsf{id}_{TX}$, and (iii) for any $X, Y, Z \in |\mathbb{J}|$, $k \in \mathbb{C}(JX, TY)$, $\ell \in \mathbb{C}(JY, TZ)$, $(\ell^* \circ k)^* = \ell^* \circ k^*$. Ordinary monads correspond to the special case $\mathbb{J} =_{\mathrm{df}} \mathbb{C}$, $J =_{\mathrm{df}} \mathsf{Id}_{\mathbb{C}}$.

Given a monoidal functor (J, e, m) between two monoidal categories $(\mathbb{J}, I, \otimes)$ and $(\mathbb{C}, I', \otimes')$, we define a strong relative monad on J as a relative monad $(T, \eta, (-)^*)$ on J that also comes with: for any $X, Y \in |\mathbb{J}|$, a map $st_{X,Y} \in \mathbb{C}(TX \otimes' JY, T(X \otimes Y))$, natural in X, Y and making the following diagrams commute:

$$\begin{array}{ccccc}
TX \otimes' I' & \xrightarrow{TX \otimes' e} & TX \otimes' JI & \xrightarrow{st_{X,I}} & T(X \otimes I) \\
\downarrow{\scriptstyle \rho'_X} & & & & \downarrow{\scriptstyle \rho_X} \\
TX & = & = & = & TX
\end{array}$$

$$\begin{array}{ccccc}
(TX \otimes' JY) \otimes' JZ & \xrightarrow{st_{X,Y} \otimes' JZ} & T(X \otimes Y) \otimes' JZ & \xrightarrow{st_{X \otimes Y, Z}} & T((X \otimes Y) \otimes Z) \\
\downarrow{\scriptstyle \alpha'_{TX,TY,TZ}} & & & & \downarrow{\scriptstyle T\alpha_{X,Y,Z}} \\
TX \otimes' (JY \otimes' JZ) & \xrightarrow[TX \otimes' m_{Y,Z}]{} & TX \otimes' J(Y \otimes Z) & \xrightarrow[st_{X, Y \otimes Z}]{} & T(X \otimes (Y \otimes Z))
\end{array}$$

$$\begin{array}{ccc}
JX \otimes' JY & \xrightarrow{m_{X,Y}} & J(X \otimes Y) \\
\downarrow{\scriptstyle \eta_X \otimes' JY} & & \downarrow{\scriptstyle \eta_{X \otimes Y}} \\
TX \otimes' JY & \xrightarrow[st_{X,Y}]{} & T(X \otimes Y)
\end{array}
\qquad
\begin{array}{ccc}
JX \otimes' JY & \xrightarrow{m_{X,Y}} & J(X \otimes Y) \\
\downarrow{\scriptstyle k \otimes' JY} & & \downarrow{\scriptstyle \ell} \\
TX' \otimes' JY & \xrightarrow[st_{X',Y}]{} & T(X' \otimes Y)
\end{array}
\Rightarrow
\begin{array}{ccc}
TX \otimes' JY & \xrightarrow{st_{X,Y}} & T(X \otimes Y) \\
\downarrow{\scriptstyle k^* \otimes' JY} & & \downarrow{\scriptstyle \ell^*} \\
TX' \otimes' JY & \xrightarrow[st_{X',Y}]{} & T(X' \otimes Y)
\end{array}$$

Arrows on a (small) category $\mathbb{J}$ are the same as relative monads on the Yoneda embedding $\mathbf{Y} \in \mathbb{J} \to [\mathbb{J}^{\mathrm{op}}, \mathbf{Set}]$, defined by $\mathbf{Y}\, Y\, X =_{\mathrm{df}} \mathbb{J}(X, Y)$. Indeed, an arrow is defined as a functor $R \in \mathbb{J}^{\mathrm{op}} \times \mathbb{J} \to \mathbf{Set}$ (i.e., an endoprofunctor on $\mathbb{J}$) together with further data satisfying certain laws, whereas a relative monad on $\mathbf{Y}$ is a functor $T \in \mathbb{J} \to [\mathbb{J}^{\mathrm{op}}, \mathbf{Set}]$, which is the same thing via $R(X,Y) =_{\mathrm{df}} T\,Y\,X$ and $T\,Y\,X =_{\mathrm{df}} R(X,Y)$.

To see that strong arrows on $\mathbb{J}$ are the same as strong relative monads on $\mathbf{Y}$, we first observe that any given monoidal structure $(I, \otimes)$ on $\mathbb{J}$ extends to a monoidal structure $(I', \otimes')$ on $[\mathbb{J}^{\mathrm{op}}, \mathbf{Set}]$ via $I'Z =_{\mathrm{df}} \mathbb{J}(Z,I)$, $(F \otimes' G)Z =_{\mathrm{df}} \int^{X,Y \in |\mathbb{J}|} \mathbb{J}(Z, X \otimes Y) \times (FX \times GY)$ (the Day convolution [2]) whereby $\mathbf{Y}$ becomes a monoidal functor. As $(T\,X \otimes' \mathbf{Y}Y)Z = \int^{X',Y' \in |\mathbb{J}|} \mathbb{J}(Z, X' \otimes Y') \times (T\,X\,X' \times \mathbb{J}(Y',Y)) \cong \int^{X' \in |\mathbb{J}|} \mathbb{J}(Z, X' \otimes Y) \times T\,X\,X'$, we have that a strength of a relative monad T on $\mathbf{Y}$ is a natural transformation with components $(st_{X,Y})_Z \in \int^{X' \in |\mathbb{J}|} \mathbb{J}(Z, X' \otimes Y) \times T\,X\,X' \to T\,(X \otimes Y)Z$. The strength of an arrow R is a (di)natural transformation with components $fst_{X',X,Y} \in R(X',X) \to R(X' \otimes Y, X \otimes Y)$, which amounts to the same.

It is viable that the view of strong arrows as strong relative monads on the Yoneda embedding may provide considerations about good ways for defining an arrow-based metalanguage. We would like to check whether these lead to the design of Lindley et al. [5].

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