An Alternative Characterization for Complete Iterativeness (Extended Abstract)

Tarmo Uustalu[1] and Varmo Vene[2]

[1] Inst. of Cybernetics, Tallinn Technical University
Akadeemia tee 21, EE-12618 Tallinn, Estonia
tarmo@cs.ioc.ee

[2] Dept. of Computer Science, University of Tartu
J. Liivi 2, EE-50409 Tartu, Estonia
varmo@cs.ut.ee

Moss [4] and Aczel, Adámek et al. [1] have recently shown that the term algebra of non-wellfounded terms in a universal-algebraic signature gives rise to a monad which is completely iterative in the sense of solvability of arbitrary systems of guarded equations. Aczel, Adámek et al. [2] have moreover shown that it is the free completely iterative monad generated by this signature.

Technically, complete iterativeness is defined for ideal monads as unique existence of an operation on morphisms of a certain type. We show that the concept admits an alternative definition where the criterion is unique existence of a natural transformation, a restriction however being that this definition can only be invoked under the existence of certain final coalgebras. We argue that reasoning about complete iterativeness can sometimes be easier resorting to the alternative definition, one of the reasons being that the diagram chase format is not ideally suited for reasoning about operations on morphisms. The alternative definition is especially useful, if the core of an argument has to be conducted in the category of endofunctors on the base category, as is the case with arguments concerning algebras of terms in binding signatures.

Ideal monads, completely iterative monads The concept of complete iterativeness is defined for monads that are ideal. A monad (T, η, μ) on $\mathcal{C}$ is said to be *ideal*, if it comes together with an endofunctor T' on $\mathcal{C}$ and natural transformations $\tau : T' \to T$, $\mu' : T' \cdot T \to T'$ such that $[\eta, \tau] : \mathsf{Id} + T' \to T$ is a natural isomorphism and

$$\begin{array}{ccc} T' \cdot T & \xrightarrow{\tau \cdot T} & T \cdot T \\ {\scriptstyle \mu'} \downarrow & & \downarrow {\scriptstyle \mu} \\ T' & \xrightarrow[\tau]{} & T \end{array}$$

An ideal monad $(T, \eta, \mu, T', \tau, \mu')$ is said to be *completely iterative*, if for any guarded equation system with unknowns in A and parameters in B, i.e., a morphism $f : A \to B + T'(A + B)$, there exists a unique morphism $h : A \to TB$

(notation solve(f)) that solves it, i.e., satisfies

$$\begin{array}{ccc}
B + T'(A+B) & \xleftarrow{\quad f \quad} & A \\
\downarrow{\scriptstyle \mathsf{inr}_{A,B} + \mathsf{id}_{T'(A+B)}} & & \\
(A+B) + T'(A+B) & & \downarrow{\scriptstyle h} \\
\downarrow{\scriptstyle [\,\eta_{A+B}, \tau_{A+B}\,]} & & \\
T(A+B) & \xrightarrow[T[\,h, \eta_B\,]]{} TTB \xrightarrow[\mu_B]{} & TB
\end{array}$$

or, which is equivalent (because of the condition relating μ and μ'),

$$\begin{array}{ccc}
B + T'(A+B) & \xleftarrow{\quad f \quad} & A \\
\downarrow{\scriptstyle \mathsf{id}_B + T'[\,h, \eta_B\,]} & & \\
B + T'TB & & \downarrow{\scriptstyle h} \\
\downarrow{\scriptstyle \mathsf{id}_B + \mu'_B} & & \\
B + T'B & \xrightarrow[{[\,\eta_B, \tau_B\,]}]{} & TB
\end{array} \tag{1}$$

The main result of [2] was that, if an endofunctor H on C is iteratable (in the sense of existence of the final $(A + H-)$-coalgebra for every C-object A), then the monad structure on the endofunctor T on C given by $TA = \nu(A + H-)$ is the free completely iterative monad generated by H. In [3], it was shown that iteratability of H is necessary in order that the free H-generated completely iterative monad exists.

An alternative definition Assume that the final $(A+T'(-+A))$-coalgebra exists for every C-object A. Set $(T^\infty A, \omega_A) = (\nu(A+T'(-+A)), \mathsf{out}_{A+T'(-+A)})$. Then one can show that $(T, \eta, \mu, T', \tau, \mu')$ is a completely iterative monad if and only if a unique natural transformation $h : T^\infty \to T$ (notation μ^∞) exists such that

$$\begin{array}{ccc}
A + T'(T^\infty A + A) & \xleftarrow{\quad \omega_A \quad} & T^\infty A \\
\downarrow{\scriptstyle \mathsf{id}_A + T'[\,h, \eta_A\,]} & & \\
A + T'TA & & \downarrow{\scriptstyle h} \\
\downarrow{\scriptstyle \mathsf{id}_A + \mu'_A} & & \\
A + T'A & \xrightarrow[{[\,\eta_A, \tau_A\,]}]{} & TA
\end{array} \tag{2}$$

The definitions of solve($-$) and μ^∞ via each other are: $\mu^\infty_A = \mathsf{solve}(\omega_A)$ and $\mathsf{solve}(f) = \mu^\infty_B \circ \mathsf{Coit}_{B+T'(-+B)}(f)$ $(f : A \to B + T'(A+B))$. By Coit, we denote coiteration: Coit_F takes a F-coalgebra structure map to the corresponding final coalgebra homomorphism.

Notice that morphisms ω_A are guarded equation systems and the condition asserts their unique solvability, so the alternative characterization replaces the requirement of unique solvability of arbitrary guarded equation systems by that of only some specific guarded equation systems which are representative of all others. This makes the relationship between μ^∞ and solve($-$) analogous to that between μ and $-^*$ (the Kleisli extension operation). While $-^*$ takes any substitution rule to the corresponding substitution function, μ delivers only those substitution functions that correspond to an identity substitution rule, since $\mu_B = \mathsf{id}_{TB}{}^\star$. Nevertheless μ determines all substitution functions, as $f^\star = \mu_B \circ Tf$ $(f : A \to TB)$.

Intuitively, the decomposition $\mathsf{solve}(f) = \mu_B^\infty \circ \mathsf{Coit}_{B+T'(-+B)}(f)$ refers to solving a guarded equation system with unknowns in A and parameters in B in two stages: first, a "quasi-solution" is calculated which assigns to the elements of A not terms over B (elements of TB), but elements of $T^\infty B$ ("quasi-terms" over B), and subsequently these quasi-terms are "flattened" into terms proper yielding the real solution. (Compare this to calculating the result of substituting a term for all occurences of a certain variable in a term by first naively replacing the variable at these occurrences by the term in question and then flattening the result into a term proper). To provide a contrast, let us note that a non-guarded equation system with unknowns from A and parameters from B is a morphism $f : A \to T(A + B)$ and any such induces a morphism $\mathsf{Coit}_{T(-+B)}(f) : A \to T^\sharp B$ where $T^\sharp B = \nu(T(- + B))$, so non-guarded equation systems with parameters from B are quasi-solvable in terms of elements of $T^\sharp B$. But for T given by $TA = \nu(A + H-)$ (the algebra of non-wellfounded terms over A in signature H) there can be no hope in general to construct a natural transformation $T^\sharp \to T$.

Applications The alternative characterization can be used to prove that the monad structure on $T = \nu(\mathsf{Id} + \mathcal{H}-)$ where $\mathcal{H} : [\mathcal{C},\mathcal{C}] \to [\mathcal{C},\mathcal{C}]$ is given by $\mathcal{H}\mathcal{X} = \mathcal{X} \times \mathcal{X} + \mathcal{X} \cdot (K_1 + \mathsf{Id}))$ (the algebra of non-wellfounded de Bruijn notations) is completely iterative by explicitly constructing a candidate for μ^∞ and checking that it verifies the required property of being the unique h satisfying (2).

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