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MENDLER-STYLE INDUCTIVE TYPES, CATEGORICALLY

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Abstract. We present a basis for a category-theoretic account of Mendler-style inductive types. The account is based on suitably defined concepts of Mendler-style algebra and algebra homomorphism; Mendler-style inductive types are identified with initial Mendler-style algebras. We use the identification to obtain a reduction of conventional inductive types to Mendler-style inductive types and a reduction in the presence of certain restricted existential types of Mendler-style inductive types to conventional inductive types.

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1. Introduction

For a category-theoretically minded programming language theorist, the term 'inductive type' normally serves as an alternative name for the familiar categorical concept of initial algebra [see, e.g., Malcolm 1990; Fokkinga 1992]. In type theory (typed lambda calculi), however, one encounters not only inductive types of this (call it conventional) sort, but also inductive types à la Mendler [see, e.g., Mendler 1991; Leivant 1990; Uustalu 1998; Matthes 1998]. This paper reports some results of an ongoing work on developing a categorical account of what we recognize as the Mendler-style conceptualization of the (apparently ambiguous) idea of inductive type. The account is based on suitably defined concepts of Mendler-style algebra and algebra homomorphism; Mendler-style inductive types are identified with initial Mendler-style algebras. We use the identification to obtain a reduction of conventional inductive types to Mendler-style inductive types and a reduction in the presence of certain restricted existential types of Mendler-style inductive types to conventional inductive types. The work has largely

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been motivated from an interest in generic functional programming and program calculation.

The Mendler-style concept of inductive type is an abstraction from Mendler's [1991] extension of system F (the pure 2nd-order simply typed lambda calculus).[1] The Mendler-style concept has two attractive features that the conventional concept has not. The first of these concerns the scope of the concept and may have a theoretical value only. There is no reason to require a base for a Mendler-style inductive type to be covariant, i.e., an endofunctor, it may well be allowed to be mixed-variant, i.e., an endodifunctor whose contravariant argument position is possibly non-dummy.[2] A conventional inductive type can only be defined for a covariant base. The other feature is of potential significance also for the programming applications point-of-view and concerns the defining properties of the concept. The equational laws characterizing a Mendler-style inductive type do not involve the morphism mapping component of the base. Those for a conventional inductive type, in contrast, manifestly refer to the morphism mapping component of the base. This feature of Mendler-style inductive types is pleasant in the light of the fact that, in type theory, a base for an inductive type is usually specified by a type constructor, i.e., an object mapping, only; a morphism mapping, called the corresponding map-function, is understood. Type constructors are represented by type schemes and map-functions by term schemes; the association function is defined by induction on type schemes.

While algebras and initial algebras are category-theoretic versions of prefixed points and least prefixed points from preorders (sets carrying a reflexive and transitive relation), Mendler-style algebras and initial Mendler-style algebras are similar versions of what might be called robustly prefixed points and least robustly prefixed points. This makes it possible to summarize the paper in preorder-theoretic terms in very few lines. Hoping that such summary might serve as a useful preview, giving some key intuitions and explaining the organization of the paper, we take the chance here.

Assume some preorder. A prefixed point (pfp) of an endofunction f on the preorder is a point c such that $fc \leq c$. If many enough subsets of the preorder have glb's, then all monotone endofunctions have a least pfp, since a glb of the pfp's of any monotone endofunction f is a least pfp of f (Tarski's theorem). A robustly prefixed point (rpfp) of an endofunction g on the preorder is a point c such that, for any point a, if $a \leq c$, then $ga \leq c$. (So, a rpfp of an endofunction is also its pfp, but the converse does not hold in general.) If many enough subsets of the preorder have glb's, then all endofunctions (monotone or not) have a least rpfp, as a glb of the rpfp's of just any endofunction g is a least rpfp of g (a robust analog of

[1] In the original conference version, [1987], of this journal paper, Mendler considered a less basic extension where, differently from the journal paper, inductive types came equipped with primitive recursors, not with iterators.

[2] This feature, apparently, must not have been realized by Mendler originally, as he required covariance. We conjecture that it was first noticed by Uustalu [1998] and Matthes [1998].

Tarski's theorem). Given a monotone endofunction f, every rpfp of f is a pfp of f and vice versa. Given an endofunction g, if, for any point c, the set $\{ga \mid a \leq c\}$ has a lub $g^{\mathrm{e}}c$, then g^{e} is a monotone function (in fact, a least monotone pointwise majorant of g) and every pfp of g^{e} is a rpfp of g and vice versa.

In connection to Mendler-style inductive types and mixed-variance, there is an important remark to be made. Generally, a Mendler-style inductive type (initial Mendler-style algebra) for a properly mixed-variant base may not be a recursive type (invariant object) for this base. [Under certain assumptions, it is guaranteed to turn out to be a conventional inductive type (initial conventional algebra) and, hence, a recursive type (invariant object) for a related covariant base, but that is something quite different, viz., a reduction of Mendler-style inductive types to conventional inductive types.] It is therefore quite likely that the results of this paper bear no relevance to the important subject of determining neat sufficient conditions for the existence of recursive types with properly mixed-variant bases and methods of constructing them.[3] On that matter, there exists an extensive body of literature [see, e.g., Wand 1979; Lehmann and Smyth 1981; Smyth and Plotkin 1982; Freyd 1990; Freyd 1991].

The paper is organized as follows. In Section 2, we recall the standard category-theoretic account of the conventional concept of inductive type. In Section 3, we present our account of the Mendler-style concept. In Section 4, we show that conventional inductive types can be reduced to Mendler-style inductive types. In Section 5, we give a categorical presentation of the concept of restricted existential type. In Section 6, we show that, in the presence of certain restricted existential types, Mendler-style inductive types can be obtained from conventional inductive types. In Section 7, we list some conclusions and questions we wish to find answers to.

Throughout the text, when doing category theory, we work with a locally small base category $\mathcal{C}$ which is not required to have any specific structure. The notation and style are largely from the constructive algorithmics community. As the main proof discipline, structured calculation is used instead of diagram chasing.

2. Conventional inductive types

A base for a conventional inductive type has to be covariant. A representative type-theoretic study of conventional inductive types treats an extension of a second-order lambda calculus with inductive types for bases represented by positive type schemes. Whether a given type scheme 'F', representing a type constructor F, is positive, is settled by induction on type schemes. If it is, then a term scheme '$\mathtt{map}_F$', representing the corresponding map-

[3] It should be noticed that such conditions typically guarantee recursive types to exist uniquely (up to isomorphism), which has the effect that the difference between inductive and coinductive types vanishes.

function $\mathtt{map}_F$, is associated to it. This has to have the property that, if the judgement $h : A \Rightarrow C$ is derivable, then so is $\mathtt{map}_F h : F\,A \Rightarrow F\,C$.

For a type constructor F given by a positive type scheme 'F', the following axioms state that μF is an inductive type with base F, $\mathtt{I}_F$ a data constructor and $\mathtt{E}_F$ an iterator.

$$\mu F : * \qquad \mu F\text{-typing}$$

$$\mathtt{I}_F : F\,\mu F \Rightarrow \mu F \qquad \mathtt{I}_F\text{-typing}$$

$$\mathtt{E}_F : \Pi C : *.(F\,C \Rightarrow C) \Rightarrow (\mu F \Rightarrow C) \qquad \mathtt{E}_F\text{-typing}$$

$$\frac{C : * \quad e : F\,C \Rightarrow C \quad c : F\,\mu F}{\mathtt{E}_F{\cdot}C{\cdot}e{\cdot}(\mathtt{I}_F{\cdot}c) = e{\cdot}(\mathtt{map}_F(\mathtt{E}_F{\cdot}C{\cdot}e){\cdot}c)} \qquad \mathtt{E}_F\text{-}\beta\text{-conversion}$$

This suggests that the following should be a correct way of adding conventional inductive types to a calculus of categorical combinators. For an endofunctor F on $\mathcal{C}$, assert the following rules to state that μF is an inductive type with base F, $\mathbf{In}_F$ a data constructor, and $(\!|-|\!)_F$ an iterator.

$$\mu F \text{ obj} \qquad \mathbf{In}_F : F\,\mu F \to \mu F \qquad (\mu F, \mathbf{In}_F)\text{-typing}$$

$$\frac{C \text{ obj} \quad \varphi : F\,C \to C}{(\!|C, \varphi|\!)_F : \mu F \to C} \qquad (\!|-|\!)_F\text{-typing}$$

$$\frac{C \text{ obj} \quad \varphi : F\,C \to C}{(\!|C, \varphi|\!)_F \circ \mathbf{In}_F = \varphi \circ F(\!|C, \varphi|\!)_F} \qquad (\!|-|\!)_F\text{-cancellation}$$

The standard categorical account of conventional inductive types is based on the categorical concepts of algebra and algebra homomorphism. An inductive type (coming together with a data constructor and an iterator) is an initial algebra.

Assume that $F : \mathcal{C} \to \mathcal{C}$ is a functor (i.e., F is an endofunctor on $\mathcal{C}$). An *F-algebra* (*algebra with signature* F) is a pair (C, φ) formed of an object C and morphism $\varphi : FC \to C$.

$$\begin{tikzcd} F\,C \arrow[d, "\varphi"'] \\ C \end{tikzcd}$$

An *F-algebra homomorphism* between two F-algebras (C, φ) and (D, ψ) is a morphism $h : C \to D$ such that $h \circ \varphi = \psi \circ Fh$.

$$\begin{tikzcd} F\,C \arrow[r, "F\,h"] \arrow[d, "\varphi"'] & F\,D \arrow[d, "\psi"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

The F-algebras and F-algebra homomorphisms form a category, $\mathbf{Alg}_F$. An *initial F-algebra* (an inductive type) is an initial object of $\mathbf{Alg}_F$, i.e., a pair $((\mu F, \mathtt{In}_F), (\!|-|\!)_F)$ formed of an F-algebra $(\mu F, \mathtt{In}_F)$ and a function $(\!|-|\!)_F$ which sends any F-algebra (C, φ) to a unique F-algebra homomorphism between $(\mu F, \mathtt{In}_F)$ and (C, φ), the *F-catamorphism* of (C, φ), satisfying these two conditions:

$$(\mu F, \mathtt{In}_F)\ \mathrm{alg}_F \qquad (\mu F, \mathtt{In}_F)\text{-typing}$$

$$\forall (C, \varphi)\ \mathrm{alg}_F.\ \forall f.\ f = (\!|\, C, \varphi \,|\!)_F \ \Leftrightarrow\ f : \mu F \to C \ \wedge\ f \circ \mathtt{In}_F = \varphi \circ F f \qquad (\!|-|\!)_F\text{-characterization}$$

The second condition is equivalent to the conjunction of the following conditions:

$$\forall (C, \varphi)\ \mathrm{alg}_F.\ (\!|\, C, \varphi \,|\!)_F : \mu F \to C \qquad (\!|-|\!)_F\text{-typing}$$

$$\forall (C, \varphi)\ \mathrm{alg}_F.\ (\!|\, C, \varphi \,|\!)_F \circ \mathtt{In}_F = \varphi \circ F (\!|\, C, \varphi \,|\!)_F \qquad (\!|-|\!)_F\text{-cancellation}$$

$$\mathtt{id}_{\mu F} = (\!|\, \mu F, \mathtt{In}_F \,|\!)_F \qquad (\!|-|\!)_F\text{-reflection}$$

$$\forall (C, \varphi)\ \mathrm{alg}_F, (D, \psi)\ \mathrm{alg}_F.\ \forall h : C \to D.\ h \circ \varphi = \psi \circ F h \ \Rightarrow\ h \circ (\!|\, C, \varphi \,|\!)_F = (\!|\, D, \psi \,|\!)_F \qquad (\!|-|\!)_F\text{-fusion}$$

Observe that, for conventional inductive types, the law of cancellation is a category-theoretic counterpart of the β-conversion rule from type theory. From the laws of reflection and fusion, well-justified type-theoretic rules for η- and permutative conversion can be inferred. This is a generality which is also true for Mendler-style inductive types and restricted existential types considered below.

EXAMPLE 1. The natural numbers can be modelled as the initial algebra $(\mu N, \mathtt{In}_N)$ of the functor $N X = 1 + X$. Write *Nat* for μN. The function $\mathit{zero} : 1 \to \mathit{Nat}$ equals $\mathtt{In}_N \circ \mathtt{inl}$, the function $\mathit{succ} : \mathit{Nat} \to \mathit{Nat}$ equals $\mathtt{In}_N \circ \mathtt{inr}$. Given a function $n : 1 \to \mathit{Nat}$, the function $\mathit{add}(n) : \mathit{Nat} \to \mathit{Nat}$ (which adds n to its argument), for instance, can be defined as the catamorphism $\mathit{add}(n) = (\!|\, \mathit{Nat}, [\, n, \mathit{succ}\,] \,|\!)_N$.

3. Mendler-style inductive types

A base for a Mendler-style inductive type may be mixed-variant. In an extension of a second-order lambda calculus with Mendler-style inductive types, any type scheme may be accepted as a representation of a base for an inductive type. For a type constructor G, the following axioms state that $\mu^{\mathrm{m}} G$ is an inductive type with base G, $\mathrm{I}^{\mathrm{m}}_G$ is a data constructor and $\mathrm{E}^{\mathrm{m}}_G$ an iterator. (Note that no condition is imposed on type schemes 'G' and no term schemes are associated to them.)

$$\mu^{\mathrm{m}}G : * \qquad \mu^{\mathrm{m}}G\text{-typing}$$

$$\mathtt{I}^{\mathrm{m}}_G : \Pi A : *.(A \Rightarrow \mu^{\mathrm{m}}G) \Rightarrow (G\,A \Rightarrow \mu^{\mathrm{m}}G) \qquad \mathtt{I}^{\mathrm{m}}_G\text{-typing}$$

$$\mathtt{E}^{\mathrm{m}}_G : \Pi C : *.(\Pi A : *.(A \Rightarrow C) \Rightarrow (G\,A \Rightarrow C)) \Rightarrow (\mu^{\mathrm{m}}G \Rightarrow C) \qquad \mathtt{E}^{\mathrm{m}}_G\text{-typing}$$

$$\frac{C : * \quad e : \Pi A : *.(A \Rightarrow C) \Rightarrow (G\,A \Rightarrow C) \quad A : * \quad d : A \Rightarrow \mu^{\mathrm{m}}G \quad c : G\,A}{\mathtt{E}^{\mathrm{m}}_G{\cdot}C{\cdot}e{\cdot}(\mathtt{I}^{\mathrm{m}}_G{\cdot}A{\cdot}d{\cdot}c) = e{\cdot}A{\cdot}(\lambda x : A.\mathtt{E}^{\mathrm{m}}_G{\cdot}C{\cdot}e{\cdot}(d{\cdot}x)){\cdot}c} \qquad \mathtt{E}^{\mathrm{m}}_G\text{-}\beta\text{-conversion}$$

From here, it is easy to infer how to add Mendler-style inductive types to a calculus of categorical combinators. For an endodifunctor G on $\mathcal{C}$, the following rules lay it down that $\mu^{\mathrm{m}}G$ is an inductive type with base G, $\mathtt{In}^{\mathrm{m}}_G$ a data constructor, and $(\!|-|\!)^{\mathrm{m}}_G$ an iterator.

$$\mu^{\mathrm{m}}G \text{ obj} \qquad \frac{A \text{ obj} \quad \alpha : A \to \mu^{\mathrm{m}}G}{\mathtt{In}^{\mathrm{m}}_G A\alpha : G(A, A) \to \mu^{\mathrm{m}}G} \qquad (\mu^{\mathrm{m}}G, \mathtt{In}^{\mathrm{m}}_G)\text{-typing}$$

$$\frac{\begin{array}{c} A \text{ obj} \quad \alpha : A \to C \\ \vdots \\ C \text{ obj} \quad \Phi A\alpha : G(A, A) \to C \end{array}}{(\!| C, \Phi |\!)^{\mathrm{m}}_G : \mu^{\mathrm{m}}G \to C} \qquad (\!|-|\!)^{\mathrm{m}}_G\text{-typing}$$

$$\frac{\begin{array}{c} A \text{ obj} \quad \alpha : A \to C \\ \vdots \\ C \text{ obj} \quad \Phi A\alpha : G(A, A) \to C \end{array} \quad A \text{ obj} \quad \alpha : A \to \mu^{\mathrm{m}}G}{(\!| C, \Phi |\!)^{\mathrm{m}}_G \circ \mathtt{In}^{\mathrm{m}}_G A\alpha = \Phi A((\!| C, \Phi |\!)^{\mathrm{m}}_G \circ \alpha)} \qquad (\!|-|\!)^{\mathrm{m}}_G\text{-cancellation}$$

The Mendler-style concept of inductive types admits a categorical account that, by its setup, mimicks the standard categorical account of the conventional concept. This account depends on concepts of Mendler-style algebra and Mendler-style algebra homomorphism. Mendler-style inductive types are identified with initial Mendler-style algebras.

For any endofunctor F on $\mathcal{C}$, define an endodifunctor $F^{\wr}$ on $\mathcal{C}$ (a padding of F with a dummy contravariant argument position) by letting $F^{\wr}(A', A) = FA$, $F^{\wr}(g', g) = Fg$. For any object C of $\mathcal{C}$ and endodifunctor G on $\mathcal{C}$, define a difunctor G/C from $\mathcal{C}$ to **Set** by letting $G/C(A', A) = \mathrm{Hom}(G(A, A'), C)$, $G/C(g', g) = \mathrm{Hom}(G(g, g'), C)$.

Assume that $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ is a functor (so G is an endodifunctor on $\mathcal{C}$). A *Mendler-style G-algebra* (*Mendler-style algebra with signature* G) is a pair (C, Φ) formed of an object C of $\mathcal{C}$ and dinatural transformation Φ between the difunctors $I^{\wr}/C$ and G/C (where I is the identity functor on $\mathcal{C}$), i.e., a dinatural function which sends any object A to a function ΦA between the sets $\mathrm{Hom}(A, C)$ and $\mathrm{Hom}(G(A, A), C)$.

$$
\begin{tikzcd}
A \text{ obj} & \Rightarrow & \mathrm{Hom}(A,C) \arrow[d, "\Phi A"'] \\
 & & \mathrm{Hom}(G(A,A),C)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \mathrm{Hom}(B,C) \arrow[dl, "\mathrm{Hom}(g,C)"'] \arrow[dr, equal] & \\
A \arrow[dd, "g"'] & & \mathrm{Hom}(A,C) \arrow[dd, "\Phi A"'] & & \mathrm{Hom}(B,C) \arrow[dd, "\Phi B"] \\
 & \Rightarrow & & & \\
B & & \mathrm{Hom}(G(A,A),C) \arrow[dr, "\mathrm{Hom}(G(g,\mathtt{id}_A),C)"'] & & \mathrm{Hom}(G(B,B),C) \arrow[dl, "\mathrm{Hom}(G(\mathtt{id}_B,g),C)"] \\
 & & & \mathrm{Hom}(G(B,A),C) &
\end{tikzcd}
$$

In other words, Φ takes objects A to functions ΦA sending morphisms $\alpha : A \to C$ to morphisms $\Phi A\alpha : G(A, A) \to C$ in some such way that the following condition is met: for any objects A, B, C and morphisms $g : A \to B$, $\alpha : A \to C$, $\beta : B \to C$, if $\alpha = \beta \circ g$, then $\Phi A\alpha \circ G(g, \mathtt{id}_A) = \Phi B\beta \circ G(\mathtt{id}_B, g)$.

$$
\begin{tikzcd}
A \arrow[d, "\alpha"'] & \Rightarrow & G(A,A) \arrow[d, "\Phi A\,\alpha"'] \\
C & & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & B \arrow[dd, "\beta"] & & & & G(B,A) \arrow[dl, "G(g,\mathtt{id}_A)"'] \arrow[dr, "G(\mathtt{id}_B,g)"] & \\
A \arrow[ur, "g"] \arrow[dr, "\alpha"'] & & \Rightarrow & & G(A,A) \arrow[dr, "\Phi A\,\alpha"'] & & G(B,B) \arrow[dl, "\Phi B\,\beta"] \\
 & C & & & & C &
\end{tikzcd}
$$

A *Mendler-style G-algebra homomorphism* between two Mendler-style G-algebras (C, Φ) and (D, Ψ) is a morphism $h : C \to D$ such that certain squares commute in **Set**.

$$A \text{ obj} \quad \Rightarrow \quad \begin{tikzcd} \mathrm{Hom}(A, C) \arrow[r, "\mathrm{Hom}(A,h)"] \arrow[d, "\Phi A"'] & \mathrm{Hom}(A, D) \arrow[d, "\Psi A"] \\ \mathrm{Hom}(G(A,A), C) \arrow[r, "\mathrm{Hom}(G(A,A),h)"'] & \mathrm{Hom}(G(A,A), D) \end{tikzcd}$$

In more low-level terms, the square says that the following holds: for any object A and morphisms $\gamma : A \to C$ and $\delta : A \to D$, if $h \circ \gamma = \delta$, then $h \circ \Phi A \gamma = \Psi A \delta$.

$$\begin{tikzcd} & A \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ C \arrow[rr, "h"'] & & D \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & G(A,A) \arrow[dl, "\Phi A \gamma"'] \arrow[dr, "\Psi A \delta"] & \\ C \arrow[rr, "h"'] & & D \end{tikzcd}$$

The Mendler-style G-algebras and G-algebra homomorphisms form a category, $\mathbf{Alg}^{\mathrm{m}}_G$. An *initial Mendler-style G-algebra* (a Mendler-style inductive type) is an initial object of $\mathbf{Alg}^{\mathrm{m}}_G$, i.e., a pair $((\mu^{\mathrm{m}} G, \mathrm{In}^{\mathrm{m}}_G), (\!| - |\!)^{\mathrm{m}}_G)$ formed of a Mendler-style G-algebra $(\mu^{\mathrm{m}} G, \mathrm{In}^{\mathrm{m}}_G)$ and a function $(\!| - |\!)^{\mathrm{m}}_G$ which sends any G-algebra (C, Φ) to a unique Mendler-style G-algebra homomorphism between $(\mu^{\mathrm{m}} G, \mathrm{In}^{\mathrm{m}}_G)$ and (C, Φ), the *Mendler-style G-catamorphism* of (C, Φ), satisfying the following conditions:

$$(\mu^{\mathrm{m}} G, \mathrm{In}^{\mathrm{m}}_G) \ \mathrm{alg}^{\mathrm{m}}_G \qquad (\mu^{\mathrm{m}} G, \mathrm{In}^{\mathrm{m}}_G)\text{-typing}$$

$$\forall (C, \Phi) \ \mathrm{alg}^{\mathrm{m}}_G. \ \forall f. \ f = (\!| C, \Phi |\!)^{\mathrm{m}}_G \ \leftrightarrow \ f : \mu G \to C \ \wedge \ (\forall A \text{ obj}, m : A \to \mu^{\mathrm{m}} G. \ f \circ \mathrm{In}_G A m = \Phi A (f \circ m)) \qquad (\!| - |\!)^{\mathrm{m}}_G\text{-characterization}$$

The second condition is equivalent to the following conditions:

$$\forall (C, \Phi) \ \mathrm{alg}^{\mathrm{m}}_G. \ (\!| C, \Phi |\!)_G : \mu G \to C \qquad (\!| - |\!)^{\mathrm{m}}_G\text{-typing}$$

$$\forall (C, \Phi) \ \mathrm{alg}^{\mathrm{m}}_G. \ (\forall A \text{ obj}, m : A \to \mu^{\mathrm{m}} G. \ (\!| C, \Phi |\!)^{\mathrm{m}}_G \circ \mathrm{In}_G A m = \Phi A ((\!| C, \Phi |\!)_G \circ m)) \qquad (\!| - |\!)^{\mathrm{m}}_G\text{-cancellation}$$

$$\mathrm{id}_{\mu G} = (\!| \mu^{\mathrm{m}} G, \mathrm{In}^{\mathrm{m}}_G |\!)^{\mathrm{m}}_G \qquad (\!| - |\!)^{\mathrm{m}}_G\text{-reflection}$$

$$\forall (C, \Phi) \ \mathrm{alg}^{\mathrm{m}}_G, (D, \Psi) \ \mathrm{alg}^{\mathrm{m}}_G. \ \forall h : C \to D. \ (\forall A \text{ obj}, \gamma : A \to C. \ h \circ \Phi A \gamma = \Psi A (h \circ \gamma)) \ \Rightarrow \ h \circ (\!| C, \Phi |\!)^{\mathrm{m}}_G = (\!| D, \Psi |\!)^{\mathrm{m}}_G \qquad (\!| - |\!)^{\mathrm{m}}_G\text{-fusion}$$

Note the fact that the morphism function part of the signature is not mentioned manifestly in the calculational laws for an initial Mendler-style algebra, it only appears in the dinaturality condition and this would in normal practice always be a "theorem for free" à la Wadler.

EXAMPLE 2. The natural numbers can also be modelled as the initial Mendler-style algebra $(\mu^{\mathrm{m}}N, \mathtt{In}^{\mathrm{m}}_N)$ of the functor $N\,X = 1 + X$. Write Nat for $\mu^{\mathrm{m}}N$. The function $\mathit{zero} : 1 \to \mathit{Nat}$ equals $\mathtt{In}^{\mathrm{m}}_N \mathit{Nat}\, \mathtt{id}_{\mathit{Nat}} \circ \mathtt{inl}$, the function $\mathit{succ} : \mathit{Nat} \to \mathit{Nat}$ equals $\mathtt{In}^{\mathrm{m}}_N \mathit{Nat}\, \mathtt{id}_{\mathit{Nat}} \circ \mathtt{inr}$. Given a function $n : 1 \to \mathit{Nat}$, the function $\mathit{add}(n) : \mathit{Nat} \to \mathit{Nat}$ (which adds n to its argument), can be defined as the Mendler-style catamorphism

$$\mathit{add}(n) = (\!|\, \mathit{Nat}, (A \text{ obj}, \alpha : A \to \mathit{Nat})[\, n, \mathit{succ} \circ \alpha\,]\,|\!)^{\mathrm{m}}_N.$$

EXAMPLE 3. Let $G Y X = (Y \Rightarrow N\,X) \times N\,X$ and write Nat' for the carrier of the initial Mendler-style G-algebra $(\mu^{\mathrm{m}}G, \mathtt{In}^{\mathrm{m}}_G)$. Assume that there exists a predecessor function $\mathit{pred}' : \mathit{Nat}' \to 1 + \mathit{Nat}'$ which satisfies the following specification: for any object A and morphism $m : A \to \mathit{Nat}'$

$$\mathit{pred}' \circ \mathtt{In}^{\mathrm{m}}_G A m = N m \circ \mathtt{snd}.$$

Then the functions $\mathit{zero}' : 1 \to \mathit{Nat}'$ and $\mathit{succ}' : \mathit{Nat}' \to \mathit{Nat}'$ can be defined as

$$\begin{array}{lcl} \mathit{zero}' & = & \mathtt{In}^{\mathrm{m}}_G \mathit{Nat}'\, \mathtt{id} \circ \langle\, \mathit{const}\, \mathit{pred}', \mathtt{inl}\,\rangle \\ \mathit{succ}' & = & \mathtt{In}^{\mathrm{m}}_G \mathit{Nat}'\, \mathtt{id} \circ \langle\, \mathit{const}\, \mathit{pred}', \mathtt{inr}\,\rangle. \end{array}$$

Now, the Fibonacci function can be defined as a Mendler-style catamorphism (there is no need to project out the result from auxiliary computation on tuples):

$$\begin{array}{lcl} \mathit{fibo} & = & (\!|\mathit{Nat}', (A \text{ obj}, \alpha : A \to \mathit{Nat}')[\mathit{one}, \\ & & \qquad\qquad [\,\mathit{one}, \mathit{add} \circ (\alpha \times \alpha)\,] \circ f] \circ \mathtt{distr}|\!)_G \\ f & = & \mathtt{distl} \circ (\mathtt{app} \times \mathtt{id}) \circ \mathtt{assoc} \circ (\mathtt{id} \times \langle\, \mathtt{id}, \mathtt{id}\,\rangle), \end{array}$$

where $\mathit{one} = \mathit{succ}' \circ \mathit{zero}' \circ !$.

4. Conventional inductive types reduced to Mendler-style inductive types

The project of this section is to show that conventional inductive types reduce to Mendler-style inductive types. To this end, we prove that, for any endofunctor F on $\mathcal{C}$, the categories $\mathbf{Alg}^{\mathrm{m}}_{F^\wr}$ and $\mathbf{Alg}_F$ are isomorphic. The proof we present is a proof from scratch. For a reader versed in category theory, the result is a simple consequence from the Yoneda lemma.

Let F be an endofunctor on $\mathcal{C}$.

DEFINITION 1. *For any Mendler-style $F^\wr$-algebra (C, Φ), define*

$$\llcorner C, \Phi \lrcorner = (C, \Phi C \mathtt{id}_C).$$

DEFINITION 2. *For any conventional F-algebra (C, φ), define*

$$\ulcorner C, \varphi \urcorner = (C, (A\,\mathrm{obj})(\gamma : A \to C)\varphi \circ F\gamma).$$

PROPOSITION 1. *If (C, Φ) is a Mendler-style $F^{\wr}$-algebra, then $\llcorner C, \Phi \lrcorner$ is a conventional F-algebra.*

PROOF. Trivial.□

PROPOSITION 2. *If (C, φ) is a conventional F-algebra, then $\ulcorner C, \varphi \urcorner$ is a Mendler-style $F^{\wr}$-algebra.*

PROOF. It has to be checked that $\ulcorner C, \varphi \urcorner$ is dinatural.

$$
\begin{array}{ll}
\rhd & \text{pick } A\,\mathrm{obj}, B\,\mathrm{obj}, g : A \to B, \beta : B \to C \\
\hline
& \ulcorner C, \varphi \urcorner A(\beta \circ g) \circ F^{\wr}(g, \mathtt{id}_A) \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
& \varphi \circ F(\beta \circ g) \circ F^{\wr}(g, \mathtt{id}_A) \\
= & \quad - F^{\wr}\text{-def} - \\
& \varphi \circ F(\beta \circ g) \circ F\mathtt{id}_A \\
= & \quad - F \text{ functorial} - \\
& \varphi \circ F\beta \circ Fg \\
= & \quad - F^{\wr}\text{-def} - \\
& \varphi \circ F\beta \circ F^{\wr}(\mathtt{id}_B, g) \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
& \ulcorner C, \varphi \urcorner B\beta \circ F^{\wr}(\mathtt{id}_B, g) \ \square
\end{array}
$$

PROPOSITION 3. *If (C, Φ) is a Mendler-style $F^{\wr}$-algebra, then*

$$\ulcorner \llcorner C, \Phi \lrcorner \urcorner = (C, \Phi).$$

PROOF.

$$
\begin{array}{ll}
\rhd & \text{pick } A\,\mathrm{obj}, \gamma : A \to C \\
\hline
& \ulcorner \llcorner C, \Phi \lrcorner \urcorner A\gamma \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
& \llcorner C, \Phi \lrcorner \circ F\gamma \\
= & \quad - \llcorner - \lrcorner\text{-def} - \\
& \Phi C\mathtt{id}_C \circ F\gamma \\
= & \quad - F^{\wr}\text{-def} - \\
& \Phi C\mathtt{id}_C \circ F^{\wr}(\mathtt{id}_C, \gamma) \\
= & \quad - \Phi \text{ dinatural} - \\
& \Phi A\gamma \circ F^{\wr}(\gamma, \mathtt{id}_A) \\
= & \quad - F^{\wr}\text{-def} - \\
& \Phi A\gamma \circ F\mathtt{id}_A \\
= & \quad - F \text{ functorial} - \\
& \Phi A\gamma \ \square
\end{array}
$$

PROPOSITION 4. *If (C, φ) is a conventional F-algebra, then*

$$\llcorner \ulcorner C, \varphi \urcorner \lrcorner = (C, \varphi).$$

PROOF.

$$\begin{array}{ll}
 & \llcorner \ulcorner C, \varphi \urcorner \lrcorner \\
= & \quad - \llcorner - \lrcorner\text{-def} - \\
 & \ulcorner C, \varphi \urcorner C \mathtt{id}_C \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
 & \varphi \circ F \mathtt{id}_C \\
= & \quad - F \text{ functorial} - \\
 & \varphi \ \square
\end{array}$$

PROPOSITION 5. *If h is a Mendler-style $F^{\wr}$-algebra homomorphism between (C, Φ) and (D, Ψ), then h is also a conventional F-algebra homomorphism between $\llcorner C, \Phi \lrcorner$ and $\llcorner D, \Psi \lrcorner$.*

PROOF.

$$\begin{array}{ll}
\rhd & \forall A \text{ obj}, \gamma : A \to C.\ \ h \circ \Phi A \gamma = \Psi A (h \circ \gamma) \\
\hline
 & h \circ \llcorner C, \Phi \lrcorner \\
= & \quad - \llcorner - \lrcorner\text{-def} - \\
 & h \circ \Phi C \mathtt{id}_C \\
= & \quad - \lhd, \text{ with } A := C, \gamma := \mathtt{id}_C - \\
 & \Psi C h \\
= & \quad - F \text{ functorial} - \\
 & \Psi C h \circ F \mathtt{id}_C \\
= & \quad - F^{\wr}\text{-def} - \\
 & \Psi C h \circ F^{\wr}(h, \mathtt{id}_C) \\
= & \quad - \Psi \text{ dinatural} - \\
 & \Psi D \mathtt{id}_D \circ F^{\wr}(\mathtt{id}_D, h) \\
= & \quad - F^{\wr}\text{-def} - \\
 & \Psi D \mathtt{id}_D \circ F h \\
= & \quad - \llcorner - \lrcorner\text{-def} - \\
 & \llcorner D, \Psi \lrcorner \circ F h \ \square
\end{array}$$

PROPOSITION 6. *If h is a conventional F-algebra homomorphism between (C, φ) and (D, ψ), then h is also a Mendler-style $F^{\wr}$-algebra homomorphism between $\ulcorner C, \varphi \urcorner$ and $\ulcorner D, \psi \urcorner$.*

PROOF.

$$
\left[\begin{array}{ll}
\triangleright & h \circ \varphi = \psi \circ F h \\
\triangleright & \text{pick } A \text{ obj}, \gamma : A \to C \\
\hline
 & h \circ \ulcorner C, \varphi \urcorner A \gamma \\
= & \qquad - \ulcorner - \urcorner\text{-def} - \\
 & h \circ \varphi \circ F \gamma \\
= & \qquad - \lhd - \\
 & \psi \circ F h \circ F \gamma \\
= & \qquad - F \text{ functorial} - \\
 & \psi \circ F (h \circ \gamma) \\
= & \qquad - \ulcorner - \urcorner\text{-def} - \\
 & \ulcorner D, \psi \urcorner A (h \circ \gamma) \ \square
\end{array}\right.
$$

These propositions tell us that there exists a functor between the categories $\mathbf{Alg}^{\mathrm{m}}_{F^{\wr}}$ and $\mathbf{Alg}_F$ and a left-and-right inverse for it.

THEOREM 1. *The categories* $\mathbf{Alg}^{\mathrm{m}}_{F^{\wr}}$ *and* $\mathbf{Alg}_F$ *are isomorphic.*

The following is now immediate:

COROLLARY 1. *If* $((\mu^{\mathrm{m}} F^{\wr}, \mathbf{In}^{\mathrm{m}}_{F^{\wr}}), (\!| - |\!)^{\mathrm{m}}_{F^{\wr}})$ *is an initial Mendler-style* $F^{\wr}$*-algebra, then* $(\llcorner \mu^{\mathrm{m}} F^{\wr}, \mathbf{In}^{\mathrm{m}}_{F^{\wr}} \lrcorner, (\!| \ulcorner - \urcorner |\!)^{\mathrm{m}}_{F^{\wr}})$ *is an initial conventional* F*-algebra.*

COROLLARY 2. *If* $((\mu F, \mathbf{In}_F), (\!| - |\!)_F)$ *is an initial conventional* F*-algebra, then* $(\ulcorner \mu F, \mathbf{In}_F \urcorner, (\!| \llcorner - \lrcorner |\!)_F)$ *is an initial Mendler-style* $F^{\wr}$*-algebra.*

5. Restricted existential types

The project opposite to that of the previous section — reducing Mendler-style inductive types to conventional inductive types — is unperformable in general. But, as we will see in Section 6, it can be carried out, if certain restricted existential types are available. Let us explain what these are.

In a second-order lambda calculus, the following axioms state that $\Sigma(H, G)$ is a restricted existential type, $\mathtt{i}_{(H,G)}$ an injector and $\mathtt{e}_{(H,G)}$ a case-operator for a constructor H of sets from types and type constructor G.

$$
\frac{}{\Sigma(H, G) : *}\ \Sigma(H,G)\text{-typing}
$$

$$
\frac{}{\mathtt{i}_{(H,G)} : \Pi A : *. \Pi a : H\, A. G\, A \Rightarrow \Sigma(H, G)}\ \mathtt{i}_{(H,G)}\text{-typing}
$$

$$
\frac{}{\mathtt{e}_{(H,G)} : \Pi C : *. (\Pi A : *. \Pi a : H\, A. G\, A \Rightarrow C) \Rightarrow (\Sigma(H, G) \Rightarrow C)}\ \mathtt{e}_{(H,G)}\text{-typing}
$$

$$
\frac{C : * \quad e : \Pi A : *. \Pi a : H\, A. G\, A \Rightarrow C \quad A : * \quad a : H\, A \quad c : G\, A}{\mathtt{e}_{(H,G)} {\cdot} C {\cdot} e {\cdot} (\mathtt{i}_{(H,G)} {\cdot} A {\cdot} a {\cdot} c) = e {\cdot} A {\cdot} a {\cdot} c}\ \mathtt{e}_{(H,G)}\text{-}\beta\text{-conversion}
$$

In a calculus of categorical combinators, the following rules state that $\Sigma(H,G)$ is a restricted existential type, in^H_G an injector, and $[-]^H_G$ a case-operator for a difunctor H from $\mathcal{C}$ to **Set** and endodifunctor G on $\mathcal{C}$.

$$\Sigma(H,G)\ \text{obj} \qquad \frac{A\ \text{obj} \quad a : H(A,A)}{\mathrm{in}^H_G A a : G(A,A) \to \Sigma(H,G)}$$

$(\Sigma(H,G), \mathrm{in}^H_G)$-typing

$$\frac{C\ \text{obj} \quad \begin{array}{c} A\ \text{obj} \quad a : H(A,A) \\ \vdots \\ \Phi A a : G(A,A) \to C \end{array}}{[C,\Phi]^H_G : \Sigma(H,G) \to C}$$

$[-]^H_G$-typing

$$\frac{C\ \text{obj} \quad \begin{array}{c} A\ \text{obj} \quad a : H(A,A) \\ \vdots \\ \Phi A a : G(A,A) \to C \end{array} \quad A\ \text{obj} \quad a : H(A,A)}{[C,\Phi]^H_G \circ \mathrm{in}^H_G A a = \Phi A a}$$

$[-]^H_G$-cancellation

Categorically, restricted existential types are restricted coends.

Assume that H is a difunctor from $\mathcal{C}$ to **Set** and G an endodifunctor on $\mathcal{C}$. An *H-restricted G-cowedge* (*cowedge from G*) is a pair (C,Φ) formed of an object C of $\mathcal{C}$ and dinatural transformation Φ between the difunctors H and G/C, i.e., a dinatural function which sends any object A of $\mathcal{C}$ to a function ΦA between the sets $H(A,A)$ and $\mathrm{Hom}(G(A,A),C)$.

$$\begin{tikzcd} A\ \text{obj} \arrow[r, phantom, "\Rightarrow"] & H(A,A) \arrow[d, "\Phi A"'] \\ & \mathrm{Hom}(G(A,A),C) \end{tikzcd}$$

$$\begin{tikzcd} & & H(B,A) \arrow[dl, "H(g,\mathrm{id}_A)"'] \arrow[dr, "H(\mathrm{id}_B,g)"] & \\ A \arrow[ddd, "g"'] & H(A,A) \arrow[ddd, "\Phi A"'] & & H(B,B) \arrow[ddd, "\Phi B"] \\ \\ & \Rightarrow & & \\ B & \mathrm{Hom}(G(A,A),C) \arrow[dr, "\mathrm{Hom}(G(g,\mathrm{id}_A),C)"'] & & \mathrm{Hom}(G(B,B),C) \arrow[dl, "\mathrm{Hom}(G(\mathrm{id}_B,g),C)"] \\ & & \mathrm{Hom}(G(B,A),C) & \end{tikzcd}$$

In other words, Φ is a function that takes objects A of $\mathcal{C}$ to functions ΦA sending elements a of $H(A,A)$ to morphisms $\Phi A a : G(A,A) \to C$ so that the following condition is met: for any objects A, B and morphism $g : A \to B$ of $\mathcal{C}$ and any element c of $H(B, A)$, it holds in $\mathcal{C}$ that $\Phi A(H(g, \mathrm{id}_A)c) \circ G(g, \mathrm{id}_A) = \Phi B(H(\mathrm{id}_B, g)c) \circ G(\mathrm{id}_B, g)$.

$$a \in H(A,A) \quad \Rightarrow \quad \begin{tikzcd} G(A,A) \arrow[d, "\Phi A a"'] \\ C \end{tikzcd}$$

$$\begin{array}{c} A \xrightarrow{g} B \\ \wedge \\ c \in H(B,A) \end{array} \quad \Rightarrow \quad \begin{tikzcd} & G(B,A) \arrow[dl, "G(g,\mathrm{id}_A)"'] \arrow[dr, "G(\mathrm{id}_B,g)"] & \\ G(A,A) \arrow[dr, "\Phi A (H(g,\mathrm{id}_A)c)"'] & & G(B,B) \arrow[dl, "\Phi B (H(\mathrm{id}_B,g)c)"] \\ & C & \end{tikzcd}$$

An *H-restricted G-cowedge homomorphism* between two H-restricted G-cowedges (C, Φ) and (D, Ψ) is a morphism $h : C \to D$ of $\mathcal{C}$ with the property that, for any object A of $\mathcal{C}$, it holds in **Set** that $\mathrm{Hom}(G(A, A), h) \circ \Phi A = \Psi A$.

$$A \text{ obj} \quad \Rightarrow \quad \begin{tikzcd} & H(A,A) \arrow[dl, "\Phi A"'] \arrow[dr, "\Psi A"] & \\ \mathrm{Hom}(G(A,A),C) \arrow[rr, "\mathrm{Hom}(G(A,A),h)"'] & & \mathrm{Hom}(G(A,A),D) \end{tikzcd}$$

This condition is equivalent to the following one: for any object A of $\mathcal{C}$ and any element a of $H(A, A)$, it is the case in $\mathcal{C}$ that $h \circ \Phi A a = \Psi A a$.

$$a \in H(A,A) \quad \Rightarrow \quad \begin{tikzcd} & G(A,A) \arrow[dl, "\Phi A a"'] \arrow[dr, "\Psi A a"] & \\ C \arrow[rr, "h"'] & & D \end{tikzcd}$$

The H-restricted G-cowedges and homomorphisms between them form a category, $\mathbf{Cow}^H_G$. An *H-restricted G-coend* (a restricted existential type) is an initial object of $\mathbf{Cow}^H_G$, i.e., a pair $((\Sigma(H, G), \mathrm{in}^H_G), [-]^H_G)$ formed of a H-restricted G-cowedge $(\Sigma(H, G), \mathrm{in}^H_G)$ and a function $[-]^H_G$ which sends any H-restricted G-cowedge (C, Φ) to a unique H-restricted G-cowedge homomorphism between $(\Sigma(H, G), \mathrm{in}^H_G)$ and (C, Φ), i.e., satisfies these conditions:

$$(\Sigma(H,G), \mathrm{in}^H_G) \ \mathrm{cow}^H_G \qquad\qquad (\Sigma(H,G), \mathrm{in}^H_G)\text{-typing}$$

$$\begin{array}{l} \forall (C,\Phi) \ \mathrm{cow}^H_G. \ \forall f. \ f = [C,\Phi]^H_G \ - \\ \quad f : \Sigma(H,G) \to C \ \wedge \ (\forall A \text{ obj}, a \in H(A,A). \ f \circ \mathrm{in}^H_G A a = \Phi A a) \end{array} \qquad [-]^H_G\text{-characterization}$$

The second condition of the two is equivalent to the conjunction of the following conditions:

$$\forall (C,\Phi)\ \mathrm{cow}^H_G.\ [\,C,\Phi\,]^H_G : \Sigma(H,G)\to C \qquad [\,-\,]^H_G\text{-typing}$$

$$\begin{array}{l}\forall (C,\Phi)\ \mathrm{cow}^H_G.\\ \quad (\forall A\ \mathrm{obj}, a\in H(A,A).\ [\,C,\Phi\,]^H_G \circ \mathbf{in}^H_G Aa = \Phi Aa)\end{array} \qquad [\,-\,]^H_G\text{-cancellation}$$

$$\mathbf{id}_{\Sigma(H,G)} = [\,\Sigma(H,G), \mathbf{in}^H_G\,]^H_G \qquad [\,-\,]^H_G\text{-reflection}$$

$$\begin{array}{l}\forall (C,\Phi)\ \mathrm{cow}^H_G, (D,\Psi)\ \mathrm{cow}^H_G.\ \forall h : C\to D.\\ \quad (\forall A\ \mathrm{obj}, a\in H(A,A).\ h\circ \Phi Aa = \Psi Aa) \ \Rightarrow\ h\circ [\,C,\Phi\,]^H_G = [\,D,\Psi\,]^H_G\end{array} \qquad [\,-\,]^H_G\text{-fusion}$$

6. Mendler-styles inductive types reduced to conventional inductive types

The necessary preparations made in the previous section, we are now in a position to construct a reduction of Mendler-style inductive types to conventional inductive types. We will obtain it in the same fashion as we obtained the reduction of conventional inductive types to Mendler-style inductive types in Section 4.

Let G be an endodifunctor on $\mathcal{C}$ such that, for any object C of $\mathcal{C}$, there exists a I^l/C-restricted G-coend $((\Sigma(I^l/C,G), \mathbf{in}^{I^l/C}_G), [\,\cdot\,]^{I^l/C}_G)$. Then, we can define the following endofunction G^e on $\mathcal{C}$:

$$\begin{array}{rcl} G^e C &=& \Sigma(I^l/C, G)\\ G^e(h : C\to D) &=& [\,\Sigma(I^l/D,G), (A\ \mathrm{obj})(\gamma : A\to C)\mathbf{in}^{I^l/D}_G A(h\circ\gamma)\,]^{I^l/C}_G . \end{array}$$

The function G^e turns out to be functorial (as one might expect), so G^e is an endofunctor on $\mathcal{C}$.

Definition 3. *Given a conventional G^e-algebra (C,φ). Define*

$$\ulcorner C,\varphi \urcorner = (C, (A\ \mathrm{obj})(\gamma : A\to C)\varphi\circ \mathbf{in}^{I^l/C}_G A\gamma).$$

Definition 4. *Given a Mendler-style G-algebra (C,Φ). Define*

$$\llcorner C,\Phi \lrcorner = (C, [\,C,\Phi\,]^{I^l/C}_G).$$

Proposition 7. *If (C,φ) is a conventional G^e-algebra, then $\ulcorner C,\varphi\urcorner$ is a Mendler-style G-algebra.*

PROOF. It has to be checked that $\ulcorner C, \varphi \urcorner$ is dinatural.

$$
\begin{array}{ll}
\rhd & \text{pick } A \text{ obj}, B \text{ obj}, g : A \to B, \beta : B \to C \\
\hline
 & \ulcorner C, \varphi \urcorner A(\beta \circ g) \circ G(g, \mathtt{id}_A) \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
 & \varphi \circ \mathtt{in}_G^{I^{\wr}/C} A(\beta \circ g) \circ G(g, \mathtt{id}_A) \\
= & \quad - \mathtt{in}_G^{I^{\wr}/C} \text{ dinatural} - \\
 & \varphi \circ \mathtt{in}_G^{I^{\wr}/C} B\beta \circ G(\mathtt{id}_B, g) \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
 & \ulcorner C, \varphi \urcorner B\beta \circ G(\mathtt{id}_B, g) \ \square
\end{array}
$$

PROPOSITION 8. *If (C, Φ) is a Mendler-style G-algebra, then $\llcorner C, \Phi \lrcorner$ is a conventional G^{e}-algebra.*

PROOF. Trivial.$\square$

PROPOSITION 9. *If (C, φ) is a conventional G^{e}-algebra, then*

$$\llcorner \ulcorner C, \varphi \urcorner \lrcorner = (C, \varphi).$$

PROOF.

$$
\begin{array}{ll}
 & \llcorner \ulcorner C, \varphi \urcorner \lrcorner \\
= & \quad - \llcorner - \lrcorner\text{-def} - \\
 & [C, \ulcorner C, \varphi \urcorner]_G^{I^{\wr}/C} \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
 & [C, (A \text{ obj})(\gamma : A \to C)\varphi \circ \mathtt{in}_G^{I^{\wr}/C} A\gamma]_G^{I^{\wr}/C} \\
= & \quad - \text{case fusion} - \\
 & \varphi \circ [G^{\mathrm{e}} C, \mathtt{in}_G^{I^{\wr}/C}]_G^{I^{\wr}/C} \\
= & \quad - \text{case reflection} - \\
 & \varphi \ \square
\end{array}
$$

PROPOSITION 10. *If (C, Φ) is a Mendler-style G-algebra, then*

$$\ulcorner \llcorner C, \Phi \lrcorner \urcorner = (C, \Phi).$$

PROOF.

$$
\begin{array}{ll}
\rhd & \text{pick } A \text{ obj}, \gamma : A \to C \\
\hline
 & \ulcorner \llcorner C, \Phi \lrcorner \urcorner A\gamma \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
 & \llcorner C, \Phi \lrcorner \circ \mathtt{in}_G^{I^{\wr}/C} A\gamma \\
= & \quad - \llcorner - \lrcorner\text{-def} - \\
 & [C, \Phi]_G^{I^{\wr}/C} \circ \mathtt{in}_G^{I^{\wr}/C} A\gamma \\
= & \quad - \text{case cancellation} - \\
 & \Phi A\gamma \ \square
\end{array}
$$

Proposition 11. *If h is a conventional G^{e}-algebra homomorphism between (C, φ) and (D, ψ), then h is also a Mendler-style G-algebra homomorphism between $\ulcorner C, \varphi \urcorner$ and $\ulcorner D, \psi \urcorner$.*

Proof.

$$
\begin{array}{ll}
\rhd & h \circ \varphi = \psi \circ G^{\mathrm{e}} h \\
\rhd & \text{pick } A \text{ obj}, \gamma : A \to C \\
\hline
 & h \circ \ulcorner C, \varphi \urcorner A \gamma \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
 & h \circ \varphi \circ \mathbf{in}_G^{\Gamma'/C} A \gamma \\
= & \quad - \lhd - \\
 & \psi \circ G^{\mathrm{e}} h \circ \mathbf{in}_G^{\Gamma'/C} A \gamma \\
= & \quad - G^{\mathrm{e}}\text{-def} - \\
 & \psi \circ [\, G^{\mathrm{e}} D, (A \text{ obj})(\gamma : A \to C) \mathbf{in}_G^{\Gamma'/D} A (h \circ \gamma) \,]_G^{\Gamma'/C} \circ \mathbf{in}_G^{\Gamma'/C} A \gamma \\
= & \quad - \text{case cancellation} - \\
 & \psi \circ \mathbf{in}_G^{\Gamma'/D} A (h \circ \gamma) \\
= & \quad - \ulcorner - \urcorner\text{-def} - \\
 & \ulcorner D, \psi \urcorner A (h \circ \gamma) \; \square
\end{array}
$$

Proposition 12. *If h is a Mendler-style G-algebra homomorphism between (C, Φ) and (D, Ψ), then h is also a conventional G^{e}-algebra homomorphism between $\llcorner C, \Phi \lrcorner$ and $\llcorner D, \Psi \lrcorner$.*

Proof.

$$
\begin{array}{ll}
\rhd & \forall A \text{ obj}, \gamma : A \to C.\;\; h \circ \Phi A \gamma = \Psi A (h \circ \gamma) \\
\hline
 & h \circ \llcorner C, \Phi \lrcorner \\
= & \quad - \llcorner - \lrcorner\text{-def} - \\
 & h \circ [\, C, \Phi \,]_G^{\Gamma'/C} \\
= & \quad - \text{case fusion} - \\
 & [\, D, (A \text{ obj})(\gamma : A \to C) h \circ \Phi A \gamma \,]_G^{\Gamma'/C} \\
= & \quad - \lhd, \text{ with } A := A, \gamma := \gamma - \\
 & [\, D, (A \text{ obj})(\gamma : A \to C) \Psi A (h \circ \gamma) \,]_G^{\Gamma'/C} \\
= & \quad - \text{case cancellation} - \\
 & [\, D, (A \text{ obj})(\gamma : A \to C) [\, D, \Psi \,]_G^{\Gamma'/D} \circ \mathbf{in}_G^{\Gamma'/D} A (h \circ \gamma) \,]_G^{\Gamma'/C} \\
= & \quad - \text{case fusion} - \\
 & [\, D, \Psi \,]_G^{\Gamma'/D} \circ [\, G^{\mathrm{e}} D, (A \text{ obj})(\gamma : A \to C) \mathbf{in}_G^{\Gamma'/D} A (h \circ \gamma) \,]_G^{\Gamma'/C} \\
= & \quad - G^{\mathrm{e}}\text{-def} - \\
 & [\, D, \Psi \,]_G^{\Gamma'/D} \circ G^{\mathrm{e}} h \\
= & \quad - \llcorner - \lrcorner\text{-def} - \\
 & \llcorner D, \Psi \lrcorner \circ G^{\mathrm{e}} h \; \square
\end{array}
$$

These propositions tell us that there exists a functor between the categories $\mathbf{Alg}_{G^{\mathrm{e}}}$ and $\mathbf{Alg}_G^{\mathrm{m}}$ and a left-and-right inverse for it.

THEOREM 2. *The categories* $\mathbf{Alg}_{G^e}$ *and* $\mathbf{Alg}^{\mathrm{m}}_G$ *are isomorphic.*

From here, the following is obvious already.

COROLLARY 3. *If* $((\mu G^e, \mathrm{In}_{G^e}), (\!|-|\!)_{G^e})$ *is an initial conventional* G^e*-algebra, then* $(\ulcorner \mu G^e, \mathrm{In}_{G^e} \urcorner, (\!| \llcorner - \lrcorner |\!)_{G^e})$ *is an initial Mendler-style* G*-algebra.*

COROLLARY 4. *If* $((\mu^{\mathrm{m}} G, \mathrm{In}^{\mathrm{m}}_G), (\!|-|\!)^{\mathrm{m}}_G)$ *is an initial Mendler-style* G*-algebra, then* $(\llcorner \mu^{\mathrm{m}} G, \mathrm{In}^{\mathrm{m}}_G \lrcorner, (\!| \ulcorner - \urcorner |\!)^{\mathrm{m}}_G)$ *is an initial conventional* G^e*-algebra.*

7. Conclusions and future work

This paper is a report of an ongoing work and there are several questions that we cannot answer yet. It looks clear, however, that the concept of initial Mendler-style algebra is meaningful and that an elegant theory can be developed for it. It is also obvious that the whole development readily dualizes for Mendler-style coinductive types.

One major issue for further investigation consists in finding out the bearings on our work of the works on dinaturality and parametricity [Bainbridge *et al.* 1990; Freyd 1993; Abadi *et al.* 1993; Hasegawa 1994]. We also wish to assess the practical utility of covariant Mendler-style inductive types in generic programming and program calculation and to find out whether there are any exciting examples of properly mixed-variant Mendler-style inductive types with a functional programming interpretation.

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