Introduction to the theory of átomata

Hellis Tamm
Institute of Cybernetics
Tallinn University of Technology

Joint work with Janusz Brzozowski

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Introduction

- Nondeterministic finite automata (NFAs), introduced by Rabin and Scott in 1959, play a major role in the theory of automata and regular languages.
- For many purposes it is necessary to convert an NFA to a deterministic finite automaton (DFA).
- In particular, for every regular language there exists a unique minimal DFA.
- As well, it is possible to associate an NFA with each regular language (universal automaton, canonical residual automaton).

Our results

- We define a unique NFA — an átomaton — for every regular language.
- It has non-empty intersections of complemented and uncomplemented quotients — the atoms of the language — as its states.
- We introduce atomic NFAs, in which the right language of any state is a union of some atoms.
- This is a generalization of residual NFAs in which the right language of any state is a left quotient (which we show to be a union of atoms).
- Atomic NFAs also include átomata (where the right language of any state is an atom), trim DFAs, and the trim parts of universal automata.

Main result

- We characterize the class of NFAs for which the subset construction yields a minimal DFA.
- More specifically, we show that the subset construction applied to a trim NFA produces a minimal DFA if and only if the reverse automaton of that NFA is atomic.
- This generalizes Brzozowski's method for DFA minimization by double reversal.

Automata and languages

- An NFA is a quintuple $\mathcal{N} = (Q, \Sigma, \delta, I, F)$, where Q is a finite, non-empty set of states, Σ is a finite non-empty alphabet, $\delta : Q \times \Sigma \rightarrow 2^Q$ is the transition function, $I \subseteq Q$ is the set of initial states, and $F \subseteq Q$ is the set of final states.
- The language accepted by an NFA $\mathcal{N}$ is $L(\mathcal{N}) = \{w \in \Sigma^* \mid \delta(I, w) \cap F \neq \emptyset\}$.
- Two NFA's are equivalent if they accept the same language.
- The left and right language of a state q of $\mathcal{N}$ are $L_{I,q}(\mathcal{N}) = \{w \in \Sigma^* \mid q \in \delta(I, w)\}$, and $L_{q,F}(\mathcal{N}) = \{w \in \Sigma^* \mid \delta(q, w) \cap F \neq \emptyset\}$.
- A DFA is a quintuple $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$, with the transition function $\delta : Q \times \Sigma \rightarrow Q$, and the initial state q_0.

Quotients and the quotient DFA

- The left quotient of a language L by a word w is the language $w^{-1}L = \{x \in \Sigma^* \mid wx \in L\}$.
- The quotient DFA of a regular language L is $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$, where
  - $Q = \{w^{-1}L \mid w \in \Sigma^*\}$
  - $\delta(w^{-1}L, a) = a^{-1}(w^{-1}L)$
  - $q_0 = \varepsilon^{-1}L = L$
  - $F = \{w^{-1}L \mid \varepsilon \in w^{-1}L\}$

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  - $F = \{w^{-1}L \mid \varepsilon \in w^{-1}L\}$
- Evidently, for an NFA $\mathcal{N}$, a state q of $\mathcal{N}$, and $x \in L_{I,q}(\mathcal{N})$, $L_{q,F}(\mathcal{N}) \subseteq x^{-1}(L(\mathcal{N}))$.
- If $\mathcal{D}$ is a DFA and $x \in L_{q_0,q}(\mathcal{D})$, then $L_{q,F}(\mathcal{D}) = x^{-1}(L(\mathcal{D}))$.

Atoms

Let $L_1 = L, L_2, \ldots, L_n$ be the quotients of a regular language L.

An atom of L is any non-empty language of the form

$$A = \widetilde{L_1} \cap \widetilde{L_2} \cap \cdots \cap \widetilde{L_n},$$

where $\widetilde{L_i}$ is either L_i or $\overline{L_i}$, and at least one of the L_i is not complemented ($\overline{L_1} \cap \overline{L_2} \cap \cdots \cap \overline{L_n}$ is not an atom).

- A language has at most $2^n - 1$ atoms.
- An atom is initial if it has L_1 (rather than $\overline{L_1}$) as a term.
- An atom is final if and only if it contains ε.
- There is exactly one final atom, the atom $\widehat{L_1} \cap \widehat{L_1} \cap \cdots \cap \widehat{L_n}$, where $\widehat{L_i} = L_i$ if $\varepsilon \in L_i$, $\widehat{L_i} = \overline{L_i}$ otherwise.

Some properties of atoms

Let $A_1, \ldots, A_m$ be the atoms of L.

- Atoms are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, m\}$, $i \neq j$.
- The quotient $w^{-1}L$ of L by $w \in \Sigma^*$ is a (possibly empty) union of atoms.
- The quotient $w^{-1}A_i$ of A_i by $w \in \Sigma^*$ is a (possibly empty) union of atoms.

Átomaton

We use a one-to-one correspondence $A_i \leftrightarrow \mathbf{A}_i$ between atoms A_i of a language L and the states $\mathbf{A}_i$ of the NFA $\mathcal{A}$ defined below.

Let $L = L_1 \subseteq \Sigma^*$ be any regular language with the set of atoms $Q = \{A_1, \ldots, A_m\}$, initial set of atoms $I \subseteq Q$, and final atom A_m.

The átomaton of L is the NFA $\mathcal{A} = (\mathbf{Q}, \Sigma, \delta, \mathbf{I}, \{\mathbf{A}_m\})$, where

- $\mathbf{Q} = \{\mathbf{A}_i \mid A_i \in Q\}$,
- $\mathbf{I} = \{\mathbf{A}_i \mid A_i \in I\}$,
- $\mathbf{A}_j \in \delta(\mathbf{A}_i, a)$ if and only if $aA_j \subseteq A_i$, for all $A_i, A_j \in Q$.

Example: computing the atoms

Let L be defined by the following quotient equations:

$$L_1 = aL_2 \cup bL_1,$$
$$L_2 = aL_3 \cup bL_1 \cup \varepsilon,$$
$$L_3 = aL_3 \cup bL_2.$$

We find the atoms using these equations:

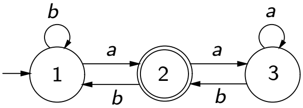
$$\begin{aligned} L_1 \cap L_2 \cap L_3 &= (aL_2 \cup bL_1) \cap (aL_3 \cup bL_1 \cup \varepsilon) \cap (aL_3 \cup bL_2) \\ &= (aL_2 \cap aL_3 \cap aL_3) \cup (bL_1 \cap bL_1 \cap bL_2) \\ &= a(L_2 \cap L_3) \cup b(L_1 \cap L_2) \\ &= a[(L_1 \cap L_2 \cap L_3) \cup (\overline{L_1} \cap L_2 \cap L_3)] \\ &\cup b[(L_1 \cap L_2 \cap L_3) \cup (L_1 \cap L_2 \cap \overline{L_3})] \end{aligned}$$

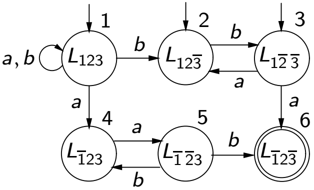
We denote $L_i \cap L_j$ by L_{ij}, $L_i \cap \overline{L_j}$ by $L_{i\bar{j}}$, etc.

Example: equations and automata

$$
\begin{aligned}
L_1 &= aL_2 \cup bL_1, \\
L_2 &= aL_3 \cup bL_1 \cup \varepsilon, \\
L_3 &= aL_3 \cup bL_2.
\end{aligned}
$$

$$
\begin{aligned}
L_{123} &= a(L_{123} \cup L_{\overline{1}23}) \cup b(L_{123} \cup L_{12\overline{3}}), \\
L_{\overline{1}23} &= aL_{\overline{1}\,\overline{2}3}, \\
L_{12\overline{3}} &= bL_{1\overline{2}\,\overline{3}}, \\
L_{\overline{1}\,\overline{2}3} &= b(L_{\overline{1}23} \cup L_{\overline{1}2\overline{3}}), \\
L_{1\overline{2}\,\overline{3}} &= a(L_{12\overline{3}} \cup L_{\overline{1}2\overline{3}}), \\
L_{\overline{1}2\overline{3}} &= \varepsilon.
\end{aligned}
$$





Some properties of átomaton

Let $A_1, \ldots, A_m$ be the atoms and let $\mathcal{A}$ be the átomaton of L.

- The right language of state $\mathbf{A}_i$ of $\mathcal{A}$ is the atom A_i, that is, $L_{\mathbf{A}_i,\{\mathbf{A}_m\}}(\mathcal{A}) = A_i$, for all $i \in \{1, \ldots, m\}$.
- The language accepted by $\mathcal{A}$ is L, that is, $L(\mathcal{A}) = L$.
- The reverse automaton $\mathcal{A}^{\mathbb{R}}$ of $\mathcal{A}$ is a minimal (incomplete) DFA for the reverse language of L.
- $\mathcal{A}$ is isomorphic to the minimal incomplete DFA of L if and only if L is bideterministic.

Atomic automata

An NFA $\mathcal{N} = (Q, \Sigma, \delta, I, F)$ is called residual, if for every state $q \in Q$, the right language $L_{q,F}(\mathcal{N})$ of q is a quotient of $L(\mathcal{N})$.

We define an NFA $\mathcal{N}$ to be atomic if for every state $q \in Q$, the right language $L_{q,F}(\mathcal{N})$ of q is a union of some atoms of $L(\mathcal{N})$.

Some examples of atomic automata:

- residual NFAs
- trim DFAs
- átomaton
- the trim part of the universal automaton

Extension of Brzozowski's Theorem

Theorem (Brzozowski, 1963). For a trim NFA $\mathcal{N}$, $\mathcal{N}^{\mathbb{D}}$ is minimal if $\mathcal{N}^{\mathbb{R}}$ is deterministic.

Brzozowski's DFA minimization algorithm:

Given any DFA $\mathcal{D}$,
1) reverse it to get $\mathcal{D}^{\mathbb{R}}$,
2) determinize $\mathcal{D}^{\mathbb{R}}$ to get $\mathcal{D}^{\mathbb{R}\mathbb{D}}$,
3) reverse $\mathcal{D}^{\mathbb{R}\mathbb{D}}$ to get $\mathcal{D}^{\mathbb{R}\mathbb{D}\mathbb{R}}$,
4) determinize $\mathcal{D}^{\mathbb{R}\mathbb{D}\mathbb{R}}$ to get $\mathcal{D}^{\mathbb{R}\mathbb{D}\mathbb{R}\mathbb{D}}$.

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4) determinize $\mathcal{D}^{\mathbb{R}\mathbb{D}\mathbb{R}}$ to get $\mathcal{D}^{\mathbb{R}\mathbb{D}\mathbb{R}\mathbb{D}}$.

Our generalization:

Theorem. For a trim NFA $\mathcal{N}$, $\mathcal{N}^{\mathbb{D}}$ is minimal if and only if $\mathcal{N}^{\mathbb{R}}$ is atomic.

Conclusions

- We have introduced a natural set of languages – the atoms – that are defined by every regular language.
- We defined a unique NFA for every regular language, the átomaton, and related it to other known concepts.
- We introduced atomic NFAs, and showed that some known subclasses of NFAs belong to this class.
- We characterized the class of trim NFAs for which the subset construction yields a minimal DFA.