Categorical Models for Two Intuitionistic Modal Logics

Wolfgang Jeltsch

TTÜ Küberneetika Instituut

Teooriaseminar
May 10, 2012

Modal logics

- used to deal with things like possibility, belief, and time
- in this talk only time
- two new operators $\Box$ and $\Diamond$:

  $\Box\varphi$ now and at every future time, φ holds

  $\Diamond\varphi$ now or at some future time, φ holds
- later also future-only variants:

  $\Box'\varphi$ at every future time, φ holds

  $\Diamond'\varphi$ at some future time, φ holds
- $\Box$ and $\Diamond$ dual and interdefinable in classical modal logics:

$$\Box\varphi := \neg\Diamond\neg\varphi \qquad \Diamond\varphi := \neg\Box\neg\varphi$$

Kripke semantics

- used for classical modal logics
- Kripke frame:
  - set W of worlds
  - accessibility relation $R \subseteq W \times W$
- Kripke model assigns truth values to formulas for each world
- semantics of modal operators:

  $\Box\varphi$ true at w if φ is true at every w' with $(w, w') \in R$

  $\Diamond\varphi$ true at w if φ is true at some w' with $(w, w') \in R$
- Kripke frames in the temporal case:
  - worlds are times
  - accessibility relation is reflexive order of times

Concrete modal logics

- some classical logics:

K	axioms that have to hold in every modal logic
S4	additional axioms that ensure that the accessibility relation is reflexive and transitive

- some intuitionistic logics and their categorical models:

IK	BCCCs with additional structure for modeling $\Box$ and $\Diamond$
CS4/IS4	additional structure that corresponds to reflexivity and transitivity of accessibility relations in the classical case

This talk

- categorical models of intuitionistic S4 based on categorical models of CS4 and IS4
- categorical models for an intuitionistic temporal logic:
  - additional structure for modeling future-only operators
  - additional structure that corresponds to totality of accessibility orders in the classical case

Basic structure

- remember:
  - objects model propositions
  - if objects A and B model propositions φ and ψ, morphisms $f : A \to B$ model proofs of $\varphi \vdash \psi$
- BCCCs as models of intuitionistic propositional logic:

$$1 \mathrel{\hat{=}} \top \qquad \times \mathrel{\hat{=}} \wedge \qquad 0 \mathrel{\hat{=}} \bot \qquad + \mathrel{\hat{=}} \vee \qquad \to \mathrel{\hat{=}} \Rightarrow$$

- BCCCs with additional structure as models of modal logics
- functors $\Box$ and $\Diamond$ for modeling logical operators $\Box$ and $\Diamond$
- morphism maps correspond to the following logical rules:

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi} \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}$$

- $\varphi \vdash \psi$ shall mean that at all times, φ implies ψ

Monoidal functors

- $\Box$ is a strong monoidal functor on the cartesian structure (cartesian functor):

$$\Box A \times \Box B \cong \Box(A \times B)$$
$$1 \cong \Box 1$$

- duality of $\Box$ and $\Diamond$ would mean that $\Diamond$ is a strong monoidal functor on the cocartesian structure:

$$\Diamond(A + B) \cong \Diamond A + \Diamond B$$
$$\Diamond 0 \cong 0$$

- do not require this:
  - left-to-right transformations would transport information about the future into the present
  - would make it impossible to use temporal logic as a language for programs that run in real time (FRP)

Comonads and monads

- $\Box$ is a comonad:

$$\varepsilon_A : \Box A \to A \qquad \delta_A : \Box A \to \Box\Box A$$

- classical analog is that accessibility relations are orders:
  - type of ε corresponds to reflexivity axiom
  - type of δ corresponds to transitivity axiom
- $\Diamond$ is a monad:

$$\eta_A : A \to \Diamond A \qquad \mu_A : \Diamond\Diamond A \to \Diamond A$$

- classical analog is also that accessibility relations are orders:
  - type of η corresponds to reflexivity axiom
  - type of μ corresponds to transitivity axiom
- classically, only one reflexivity and one transitivity axiom necessary (because $\Box$ and $\Diamond$ are interdefinable)
- need both the comonad and the monad structure in the intuitionistic case

Relative tensorial strength

- $\Diamond$ is $\Box$-strong:
  - natural transformation s with

    $$s_{A,B} : \Box A \times \Diamond B \to \Diamond(\Box A \times B)$$

    exists
  - s is compatible with cartesian functor, comonad, and monad structure
- proposition corresponding to s holds automatically in classical logic (because $\Box$ and $\Diamond$ are interdefinable)

Future only

- logic with future-only operators $\Box'$ and $\Diamond'$:

$$\Box\varphi = \varphi \wedge \Box'\varphi$$
$$\Diamond\varphi = \varphi \vee \Diamond'\varphi$$

- functors $\Box'$ and $\Diamond'$ with the following properties:

$$\Box A = A \times \Box' A$$
$$\Diamond A = A + \Diamond' A$$

- $\Box'$ is an ideal comonad, and $\Diamond'$ is an ideal monad:
  - natural transformations δ' and μ' with

    $$\delta' : \Box' A \to \Box' \Box A$$
    $$\mu' : \Diamond' \Diamond A \to \Diamond' A$$

    exist
  - comonad and monad structure derived from δ' and μ'

Linear time

- classically, accessibility order must be total
- introduction of a natural transformation r with

$$r_{A,B} : \Diamond A \times \Diamond B \to \Diamond (A \odot B) \quad ,$$

where

$$A \odot B := A \times B + A \times \Diamond' B + \Diamond' A \times B$$

A nicer solution

- an operator $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ with

$$\frac{f : C \to \Diamond A \qquad g : C \to \Diamond B}{\langle\!\langle f, g \rangle\!\rangle : C \to \Diamond(A \odot B)}$$

- looks a bit like the $\langle \cdot, \cdot \rangle$-operator of a product
- require $A \odot B$ to be a product in the Kleisli category of $\Diamond$
- $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is now the $\langle \cdot, \cdot \rangle$-operator of that product
- projections:

$$\varpi_1 : A \times B + A \times \Diamond' B + \Diamond' A \times B \to A + \Diamond' A$$
$$\varpi_2 : A \times B + A \times \Diamond' B + \Diamond' A \times B \to B + \Diamond' B$$

- product axioms (in the Kleisli category) ensure that proofs of $\Diamond A$ and $\Diamond B$ can be recovered from proof of $\Diamond(A \odot B)$:

$$\mu(\Diamond \varpi_1)\langle\!\langle f, g \rangle\!\rangle = f \qquad \mu(\Diamond \varpi_2)\langle\!\langle f, g \rangle\!\rangle = g$$

- as a result, r is an isomorphism

References

Satoshi Kobayashi
Monad as Modality
Theoretical Computer Science 175 (1997), pp. 29–74

Gavin Bierman and Valeria de Paiva
On an intuitionistic modal logic
Studia Logica 65 (2000), pp. 383–416

Wolfgang Jeltsch
Towards a Common Categorical Semantics for Linear-Time Temporal Logic and Functional Reactive Programming
Electronic Notes in Theoretical Computer Science
(*to appear*)