Concrete Process Categories

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Functional reactive programming

- extension of functional programming
- supports description of temporal behavior
- two key concepts:
  - time-dependent type membership
  - special type constructors:
    - $\square$ time-varying values
    - $\diamond$ events
- Curry–Howard correspondence to temporal logic:
  - time-dependent trueness
  - special operators:
    - $\square$ will always hold
    - $\diamond$ will eventually hold

Categorical models of simply typed calculus

- models are cartesian closed categories with coproducts
- use of basic category structure:

$$\begin{array}{ccc} \text{type} & \longmapsto & \text{object} \\ \big\downarrow \; \textit{operation} & \longmapsto & \textit{morphism} \; \big\downarrow \\ \text{type} & \longmapsto & \text{object} \end{array}$$

- use of CCCC structure:

product type	$\tau_1 \times \tau_2$	$\longmapsto$	$A \times B$	product
sum type	$\tau_1 + \tau_2$	$\longmapsto$	$A + B$	coproduct
function type	$\tau_1 \to \tau_2$	$\longmapsto$	B^A	exponential
unit type	1	$\longmapsto$	1	terminal object
empty type	0	$\longmapsto$	0	initial object

Categorical models of FRP

- ingredients:

  totally ordered set $(T, \leqslant)$ time scale

  CCCC $\mathcal{B}$ simple types and functions

- product category $\mathcal{B}^T$ models FRP types and operations with indices denoting inhabitation times:

$$
\begin{array}{ccccc}
\tau_1 & \longmapsto & A(t^\dagger) & \cdots & A(t^\ddagger) \\
\big\downarrow \varphi & \longmapsto & f(t^\dagger) \big\downarrow & & \big\downarrow f(t^\ddagger) \\
\tau_2 & \longmapsto & B(t^\dagger) & \cdots & B(t^\ddagger)
\end{array}
$$

Meanings of FRP type constructors

- general picture:

$$\text{simple type constructors} \longmapsto \text{CCCC structure of } \mathcal{B}^T$$
$$\text{type constructors } \Box \text{ and } \Diamond \longmapsto \text{functors } \Box \text{ and } \Diamond$$

- CCCC structure of $\mathcal{B}^T$ from CCCC structure of $\mathcal{B}$ with operations working pointwise
- functors $\Box$ and $\Diamond$ defined as follows:

$$(\Box A)(t) = \prod_{t' \in [t,\infty)} A(t')$$
$$(\Diamond A)(t) = \coprod_{t' \in [t,\infty)} A(t')$$

From “until” to processes

- more temporal operators from linear-time temporal logic:
  - ▷ strong “until”
  - ▶ weak “until”
- semantics given by functors ▷ and ▶:

$$(A \rhd B)(t) = \coprod_{t' \in [t,\infty)} \left(\prod_{t'' \in [t,t')} A(t'') \times B(t') \right)$$

$$(A \blacktriangleright B)(t) = (A \rhd B)(t) + \prod_{t' \in [t,\infty)} A(t')$$

- FRP analogs of “until” proofs are processes:
  - normally finite-length time-varying value plus terminal event
  - in the case of ▶ also nontermination possible

Applications of processes

- stereo playback with different guarantees:

  $(\mathbb{R} \times \mathbb{R}) \blacktriangleright 1$ none

  $(\mathbb{R} \times \mathbb{R}) \triangleright 1$ termination

  $(\mathbb{R} \times \mathbb{R}) \blacktriangleright 0$ nontermination

- stereo playback with additional information:

  $(\mathbb{R} \times \mathbb{R}) \blacktriangleright (1 + 1)$ reason of termination (end of track vs. abort)

- alternating stereo/mono playback with different guarantees:

  $\nu\sigma \,.\, (\mathbb{R} \times \mathbb{R}) \blacktriangleright \mathbb{R} \blacktriangleright \sigma$ nontermination

  $\nu\sigma \,.\, (\mathbb{R} \times \mathbb{R}) \triangleright \mathbb{R} \triangleright \sigma$ switch, nontermination

  $\nu\sigma \,.\, (\mathbb{R} \times \mathbb{R}) \blacktriangleright (1 + \mathbb{R} \blacktriangleright (1 + \sigma))$ none

  $\nu\sigma \,.\, (\mathbb{R} \times \mathbb{R}) \triangleright (1 + \mathbb{R} \triangleright (1 + \sigma))$ switch

  $\mu\sigma \,.\, (\mathbb{R} \times \mathbb{R}) \triangleright (1 + \mathbb{R} \triangleright (1 + \sigma))$ termination

Processes as the core concept of FRP

- introduction of processes increases expressiveness
- processes cover time-varying values and events as special cases:

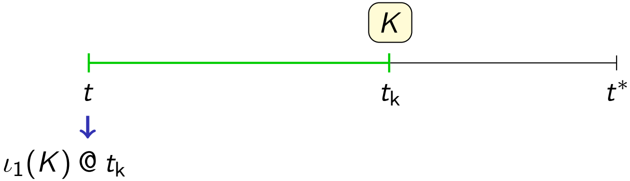
$$\Box A \cong A \blacktriangleright 0$$

$$\Diamond A \cong 1 \triangleright A$$

An example program component

- looks for the next key press up to a certain timeout
- emits a value of type $\Diamond(\mathsf{Key} + 1)$ when it starts:

  Case 1 key press before timeout:



  Case 2 no key press before timeout:

A noncausal operation

- hypothetical polymorphic operation d from $\Diamond(\tau_1 + \tau_2)$ to $\Diamond\tau_1 + \Diamond\tau_2$:

$$\iota_1(x) \mathbin{@} t' \mapsto \iota_1(x \mathbin{@} t')$$
$$\iota_2(y) \mathbin{@} t' \mapsto \iota_2(y \mathbin{@} t')$$

- applying d to the output of the key press listener gives value of type $\Diamond\mathrm{Key} + \Diamond 1$:

key press before timeout	$\iota_1(K \mathbin{@} t_{\mathrm{k}})$
no key press before timeout	$\iota_2(\mathrm{tt} \mathbin{@} t^*)$

- tells us immediately if the user will press a key before the timeout
- so d cannot exist

Semantics allow for noncausal operations

- polymorphic operations from $\Diamond(\tau_1 + \tau_2)$ to $\Diamond\tau_1 + \Diamond\tau_2$ modeled by natural transformations τ with

$$\tau_{A,B} : \Diamond(A + B) \to \Diamond A + \Diamond B$$

- there is such a τ (which is even an isomorphism):

$$\coprod_{t' \geqslant t} \Big(A(t') + B(t')\Big) \cong \coprod_{t' \geqslant t} A(t') + \coprod_{t' \geqslant t} B(t')$$

- reason:

  semantics do not deal with time-dependent knowledge about values

Knowledge-aware semantics

- replace category $\mathcal{B}^T$ by category $\mathcal{B}^I$ where

$$I = \{(t, t_o) \in T \times T \mid t \leqslant t_o\}$$

- dealing with knowledge at t_o:

$$\begin{array}{ccc}
\text{knowledge type} & \longmapsto & A(t, t_o) \\
\big\downarrow \quad \textit{knowledge transformation} & \longmapsto & f(t, t_o) \big\downarrow \\
\text{knowledge type} & \longmapsto & B(t, t_o)
\end{array}$$

- $(A \blacktriangleright B)(t, t_o)$ defined as follows:

$$\coprod_{t' \in [t, t_o]} \left(\prod_{t'' \in [t, t')} A(t'', t_o) \times B(t', t_o) \right) + \prod_{t' \in [t, t_o]} A(t', t_o)$$

Compatibility of knowledge transformations

- knowledge transformations may be incompatible
- extend set I to category $\mathcal{I}$ by adding morphisms

$$(t, t_o, t'_o) : (t, t'_o) \to (t, t_o)$$

  for $t \leqslant t_o \leqslant t'_o$
- replace product category $\mathcal{B}^I$ by functor category $\mathcal{B}^{\mathcal{I}}$
- objects $A(t, t_o, t'_o)$ model knowledge reduction
- morphisms of $\mathcal{B}^{\mathcal{I}}$ are natural transformations
- means that knowledge transformations are compatible:

$$\begin{array}{ccc}
A(t, t_o) & \xleftarrow{\;A(t, t_o, t'_o)\;} & A(t, t'_o) \\
\Big\downarrow{\scriptstyle f_{(t,t_o)}} & & \Big\downarrow{\scriptstyle f_{(t,t'_o)}} \\
B(t, t_o) & \xleftarrow[\;B(t, t_o, t'_o)\;]{} & B(t, t'_o)
\end{array}$$

Upper bounds for occurrence times

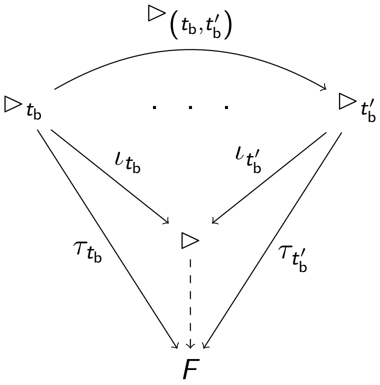
- definition of functor $\triangleright$ not directly possible
- introduction of new functor $\triangleright_{\_} : \mathcal{T} \to \left(\mathcal{B}^{\mathcal{I}}\right)^{\mathcal{B}^{\mathcal{I}} \times \mathcal{B}^{\mathcal{I}}}$ where $\mathcal{T}$ is the category of $(T, \leqslant)$
- $\triangleright_{t_b}$ models a process type constructor with upper bound t_b for termination time
- $(A \triangleright_{t_b} B)(t, t_o)$ defined as follows:

$$\begin{cases} 0 & \text{if } t_b < t \\ \coprod_{t' \in [t, t_b]} \left(\prod_{t'' \in [t, t')} A(t'', t_o) \times B(t', t_o) \right) & \text{if } t \leqslant t_b \leqslant t_o \\ (A \blacktriangleright B)(t, t_o) & \text{it } t_o < t_b \end{cases}$$

- $\triangleright_{(t_b, t'_b)}$ models type conversion

Definition of the $\triangleright$-functor

- type constructor $\triangleright$ is the least upper bound of all $\triangleright_{t_b}$-constructors
- functor $\triangleright$ must be a colimit of the functor $\triangleright_{\_}$:



The shape of the $\triangleright$-functor

Theorem

If $(T, \leqslant)$ *has a maximum* t_{max}*, then* $\triangleright \cong \triangleright_{t_{\mathrm{max}}}$.

Theorem

If $(T, \leqslant)$ *has no maximum, then* $\triangleright \cong \blacktriangleright$.

Causality ensured

Theorem

There are categorical models that do not contain any natural transformation τ *with*

$$\tau_{A,B} : \Diamond(A + B) \to \Diamond A + \Diamond B \quad .$$

Conclusions

- processes:
  - result of extending the Curry–Howard correspondence between FRP and temporal logic to cover "until" operators
  - make FRP more expressive
  - generalize time-varying values and events nicely
- knowledge-aware categorical models:
  - express causality of FRP operations
  - cannot express liveness constraint of $\triangleright$ for unbounded time
- ultimate goal is an axiomatic semantics with the following properties:
  - expresses causality
  - expresses liveness constraint of $\triangleright$ generally
  - covers concrete process categories as a special case